

MOSER-TRUDINGER TYPE INEQUALITIES FOR COMPLEX MONGE-AMPÈRE OPERATORS AND AUBIN'S "HYPOTHÈSE FONDAMENTALE"

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ABSTRACT. We prove Aubin's "Hypothèse fondamentale" concerning the existence of Moser-Trudinger type inequalities on any integral compact Kähler manifold X . In the case of the anti-canonical class on a Fano manifold the constants in the inequalities are shown to only depend on the dimension of X (but there are counterexamples to the precise value proposed by Aubin). In the different setting of pseudoconvex domains in complex space we also obtain a quasi-sharp version of the inequalities and relate it to Brezis-Merle type inequalities. The inequalities are shown to be sharp for S^1 -invariant functions on the unit-ball. We give applications to existence and blow-up of solutions to complex Monge-Ampère equations of mean field (Liouville) type.

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1. INTRODUCTION

As shown by Trudinger in the seminal work [52] there is a limiting exponential version of the critical Sobolev inequalities which, in the case of the plane, may be formulated as the existence of positive constants c and C such that

$$(1.1) \quad \int_{\Omega} e^{c \left(\frac{u}{\|\nabla u\|_{\Omega}} \right)^2} dV \leq C$$

for any, say smooth, function u vanishing on the boundary of a domain Ω in \mathbb{R}^2 . Motivated by the Nirenberg problem for constructing conformal metrics on a real surface with prescribed positive curvature, Moser [45] obtained the sharp constant $c = 4\pi$ in Trudinger's inequality 1.1. The relation to the Nirenberg problem appears in the following consequence of the previous inequality:

$$(1.2) \quad \log \int_{\Omega} e^{-u} dV \leq A \|\nabla u\|_{\Omega}^2 + B$$

Here e^{-u} plays the role of the conformal factor of a metric on Ω . As shown by Moser the inequalities also hold when the domain Ω is replaced by the two-sphere - which is the setting for the Nirenberg problem - and then the extremals u of the inequality correspond to metrics g_u with constant positive curvature

(with $A = 1/16\pi$, the sharp constant). Conversely, the latter inequality 1.2, with the sharp constant, implies an inequality of the form 1.1, but only with quasi-sharp constants, i.e. the two inequalities are equivalent “modulo ϵ ”.

There has been a wealth of work on extending Moser-Trudinger inequalities in various directions in real analysis and conformal geometry (see for example [5, 31, 4, 32] and references therein). However, the present paper is concerned with a different *complex* variant of these inequalities first proposed by Aubin [3], motivated by the existence problem for Kähler-Einstein metrics with positive curvature on complex (Fano) manifolds (see also [29, 30, 46]). More precisely, we will consider two different settings: (1) compact complex (Kähler) manifolds (without boundary) and (2) pseudoconvex domains in \mathbb{C}^n . A characteristic feature of the complex setting (when $n > 1$) is that it is considerably more non-linear than the real one. Indeed, the corresponding inequalities (see below) only hold for a *convex* subspace \mathcal{H}_0 of functions u and moreover the Laplacian Δ appearing in the Dirichlet energy $\|\nabla u\|_\Omega^2 (= \int_\Omega -u\Delta u dV)$ has to be replaced by fully *non-linear* complex Monge-Ampère operators. Moreover, in the compact setting (1) the space \mathcal{H}_0 is not even a cone and the corresponding Monge-Ampère operator is not n -homogeneous (in contrast to the setting (2)).

1.1. Statement of the main results.

1.1.1. *The setting of a compact Kähler manifold.* Let (X, ω) be a compact Kähler manifold without boundary of complex dimension n and recall that a smooth function u on X is called a *Kähler potential* if

$$\omega_u := \omega + \frac{i}{2\pi} \partial\bar{\partial}u := \omega + dd^c u > 0,$$

i.e. ω_u is a Kähler metric in the cohomology class $[\omega] \in H^2(X, \mathbb{R})$. We will denote by $\mathcal{H}_0(X, \omega)$ the convex space of all such u normalized so that $\sup_X u = 0$ and we will consider the following well-known functional on $\mathcal{H}_0(X, \omega)$:

$$(1.3) \quad \mathcal{E}_\omega(u) := \frac{1}{(n+1)!} \sum_{j=0}^n \int_X u(\omega_u)^j \wedge (\omega)^{n-j}$$

that we will refer to as (minus) the *Monge-Ampère energy*.

Theorem 1.1. *Let (X, ω) be a Kähler manifold such that $[\omega] \in H^2(X, \mathbb{Z}) \otimes \mathbb{R}$. Then the following Moser-Trudinger type inequality holds for any function u in $\mathcal{H}_0(X, \omega)$ and positive number k :*

$$(1.4) \quad \log \int_X e^{-ku} dV \leq Ak^{n+1}(-\mathcal{E}_\omega(u)) + B$$

for some positive constants A and B (given a volume form dV). More precisely, the constant A may be replaced by $(1 + C_1/k)$ and B by $(1 + C_2/k)$ for certain invariants C_1 and C_2 of ω (see 2.6).

The first part of the theorem establishes a conjecture of Aubin (called “Hypothèse fondamentale” in [3]) under the assumption that the class $[\omega]$ be integral. The inequalities 1.1 are equivalent to the existence of positive constants c and C such that

$$(1.5) \quad \int_X e^{c\left(\frac{-u}{(-\mathcal{E}(u))^{1/(n+1)}}\right)^{(n+1)/n}} dV \leq C,$$

providing a variant of Trudinger’s inequality 1.1 in the Kähler setting. It appears to be new even in the case of two-dimensional projective space. In particular

we deduce the following Sobolev type inequalities of independent interest: for any u in $\mathcal{H}_0(X, \omega)$

$$(1.6) \quad \|u\|_{L^p(X)}^{n+1} \leq Cp^n(-\mathcal{E}_\omega(u))$$

for all p in $]1, \infty[$, for some constant C only depending on ω .

The starting point of the proof of the previous theorem is the basic fact that, in the integral case when $[\omega] \in H^2(X, \mathbb{Z})$, the space $k\mathcal{H}_0(X, \omega)$ may be identified (mod \mathbb{R}) with the space $\mathcal{H}(kL)$ of all positively curved metrics on the k th tensor of an ample line bundle $L \rightarrow X$ with Chern class $c_1(L) = [\omega]$. The proof then exploits convexity properties along geodesics of certain functionals on the space $\mathcal{H}(L)$ equipped with the Mabuchi metric (see section 1.3 for an outline of the proof).

As pointed out above Aubin's main motivation for his conjecture came from the existence problem for positively curved Kähler-Einstein metrics on a Fano manifold where the Kähler class $[\omega]$ is the integral class $c_1(-K_X)$, i.e. the first Chern class of the anti-canonical line bundle $-K_X$ of X . In this setting, which we will refer to as the Fano setting, he also conjectured an explicit optimal value for A which only depends on the dimension n of the Fano manifold. However, as explained in section 6 there is a simple counter-example to the explicit value proposed by Aubin. Still, combining our arguments with previous work on finiteness properties of Fano manifolds [51, 41, 19] we deduce the following partial confirmation of Aubin's latter conjecture:

Theorem 1.2. *When X is an n -dimensional Fano manifold and $[\omega]$ is the anti-canonical class the constant A can be taken to only depend on n (if B is allowed to depend on k).*

Coming back to the general setting in Theorem 1.1 we point out that the Moser-Trudinger type inequalities there will be shown to hold as long as $\omega = c_1(L)$ for L semi-positive and such that the adjoint bundles $kL + K_X$ are base point free for all sufficiently large positive integers k . Using this and L^2 -estimates for $\bar{\partial}$ we deduce the following "qualitative" Moser-Trudinger type inequality in a degenerate setting:

Corollary 1.3. *Let ω be a semi-positive form on compact complex manifold X such that $[\omega] = c_1(L)$ with L semi-positive and big (i.e. $\int_X c(L)^n > 0$). Then u has vanishing Lelong numbers, i.e. any negative multiple of an ω -psh function u with finite energy is exponentially integrable. More precisely, if $\mathcal{E}_\omega(u) \geq -C$ and $\sup_X u = 0$ then*

$$\int_X e^{-ku} dV \leq C_k,$$

where the constant C_k only depends on C and k .

The notion of finite energy is recalled in the beginning of section 2.4. In the strictly positive case the previous corollary is due to Guedj-Zeriahi [34] and proved using pluripotential theory. The present extension to the semi-positive case was motivated by the work [11] where it is used in the construction of Kähler-Einstein metrics on singular Fano varieties.

1.1.2. The setting of a pseudoconvex domain in \mathbb{C}^n . Let now Ω be a pseudoconvex domain in \mathbb{C}^n with smooth boundary (for example the unitball) and set $\omega := 0$. In this setting we let $\mathcal{H}_0(\Omega)$ be the convex cone of all smooth plurisubharmonic functions, i.e. $dd^c u \geq 0$, vanishing on the boundary $\partial\Omega$. Then the $n+1$ -homogeneous functional

$$(1.7) \quad n!\mathcal{E}_0(u) = \frac{1}{(n+1)} \int_\Omega u (dd^c u)^n$$

is the usual generalization to \mathbb{C}^n of (minus) the squared Dirichlet norm in the unit disc. In the paper [3] Aubin claims that the conjectured inequality holds in the setting of the unit-ball in \mathbb{C}^n , but it appears that he only proved this under radial symmetry ([4], Cor 8.3 in) and in fact with a non-optimal constant (as explained in section 6). Assuming only circular symmetry, i.e. invariance under the diagonal S^1 -action on \mathbb{C}^n , our method of proof of Theorem 1.1 also yields the following generalization of Moser's inequality on the disc:

Theorem 1.4. *The following Moser-Trudinger inequality holds for any S^1 -invariant function in $\mathcal{H}_0(\mathcal{B})$, where \mathcal{B} is the unit-ball in \mathbb{C}^n :*

$$(1.8) \quad \log \int_{\mathcal{B}} e^{-u} dV \leq \frac{1}{(n+1)(n+1)} \int_{\Omega} (-u)(dd^c u)^n + C_n$$

for a constant C_n . Moreover the multiplicative constant in the inequality is sharp.

Note that the sharp constant in 1.8 coincides with the well-known one in the Fano setting when $X = \mathbb{P}^n$ and $k = 1$ (and our proof shows that this is no coincidence). We conjecture that the symmetry assumption in the previous theorem may be removed. In this direction we will prove the following quasi-sharp Moser-Trudinger inequality for a general pseudoconvex domain (or more generally a hyperconvex one):

Theorem 1.5. *Let Ω be a pseudoconvex domain in \mathbb{C}^n with smooth boundary. Then, for any $\delta > 0$ there is a constant C such*

$$(1.9) \quad \log \int_{\Omega} e^{-u} dV \leq \frac{1+\delta}{(n+1)(n+1)} \int_{\Omega} (-u)(dd^c u)^n - (n-1) \log \delta + C_n$$

for any function u in $\mathcal{H}_0(\Omega)$. Moreover, for any domain Ω the limiting multiplicative constant $\frac{1}{(n+1)(n+1)}$ is sharp. In particular, for any $\delta > 0$ there is a constant C_δ such that

$$\int_{\Omega} e^{(1-\delta)n(-u)^{(n+1)/n}} dV \leq C_\delta$$

for any u in $\mathcal{H}_0(\Omega)$ such that $\int_{\Omega} (-u)(dd^c u)^n = 1$.

The proof of the latter theorem is completely different than the previous one. The starting point is the observation that if the sharp Moser-Trudinger inequality holds in dimension $n-1$ then so does the following sharp *Brezis-Merle type inequality*:

$$(1.10) \quad \int_{\Omega} e^{-u} dV \leq A \left(1 - \frac{1}{n^n} \mathcal{M}(u) \right)^{-1}$$

for any u in $\mathcal{H}_0(\Omega)$ such that $\mathcal{M}(u)^{1/n} < n$, where $\mathcal{M}(u)$ is the total Monge-Ampère mass of u :

$$(1.11) \quad \mathcal{M}(u) := \int_{\Omega} (dd^c u)^n$$

(see [17] for the case when $n = 1$ and its relation to blow-up analysis of PDEs). We then show that, conversely a quasi-sharp version of the Brezis-Merle inequality in dimension n implies the quasi-sharp Moser-Trudinger inequality above in the same dimension n and Theorem 1.5 then follows directly from induction over n . More precisely, the induction argument gives the following quasi-sharp version of the conjectural Brezis-Merle type inequality above.

Theorem 1.6. *Let Ω be a pseudoconvex domain in \mathbb{C}^n with smooth boundary, where $n > 1$. Then there is a constant A such*

$$(1.12) \quad \int_{\Omega} e^{-u} dV \leq A \left(1 - \frac{1}{n^n} \mathcal{M}(u) \right)^{-(n-1)}$$

for any function in $\mathcal{H}_0(\Omega)$ such that $\mathcal{M}(u)^{1/n} < n$.

In particular, this proves the sharp inequality in the case when $n = 2$.

1.1.3. Applications to Monge-Ampère equations. In section 7 we consider the problem of finding extremals for Moser-Trudinger type functionals that are parametrized by the multiplicative constants in the corresponding inequalities. In particular, we obtain solutions to the Euler-Lagrange equations for these functionals which are Monge-Ampère equations with exponential non-linearities. In the settings of domains we obtain the following

Theorem 1.7. *Let Ω be a pseudoconvex domain in \mathbb{C}^n and assume that $a < (n+1)^n$. Then there exists $u \in \mathcal{C}^0(\bar{\Omega})$ solving the equation*

$$(dd^c u)^n = a \frac{e^{-u} dV}{\int_{\Omega} e^{-u} dV} \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega$$

such that u optimizes the corresponding Moser-Trudinger type functional.

Here $(dd^c u)^n$ refers to the usual notion of Monge-Ampère measure in pluripotential theory introduced by Bedford-Taylor. In the case when $n = 1$ these equations are often called *mean field equations* in the literature, as they appear in a statistical mechanical context [18, 38] (see section 7.3). We also establish a “concentration/compactness” principle for the behavior of the solutions u_a above when a approaches the critical value $(n+1)^n$ (see Theorem 7.3 for the precise statement). In particular, it implies that if there is no blow-up point in the boundary of Ω , then (after passing to a subsequence) *either* u_a converges to a solution of the previous equation *or* u_a converges to a (weak) solution of the equation

$$(dd^c u)^n = (n+1)^n \delta_{z_0}, \quad u = 0 \text{ on } \partial\Omega$$

Moreover u has a minimal complex singularity exponent at z_0 [28]. It seems natural to conjecture that u coincides with $(n+1)$ times the pluricomplex Green function with a pole at z_0 [39]. This is automatically the case when $n = 1$ where there has been rather extensive work on such “concentration/compactness” principles, with various elaborations (see for example [18, 44]).

1.2. Relations to previous results.

The Kähler setting. On the two-sphere the inequality in Theorem 1.1 was first shown by Moser with the sharp constant $A = 1/2$. Subsequently, the general Riemann surface case was settled by Fontana [31] with the same sharp constant. Strictly speaking these latter inequalities were shown to hold for *any* smooth function u , under the different (but equivalent) normalization condition $\int_X u \omega = 0$. Then $-\mathcal{E}_{\omega}(u)$ coincides with the usual two-homogeneous Dirichlet energy and the growth rate with respect to k can hence be reduced, by scaling, to the case $k = 1$. It should however be emphasized that in higher dimensions this reduction argument breaks down, since the space $\mathcal{H}_0(X, \omega)$ is not preserved under scaling with positive numbers k . The sharp form of the Sobolev inequalities on the two-sphere in 1.6 was obtained by Beckner [5].

In the case when X admits a Kähler-Einstein metric the Moser-Trudinger inequality, for the anti-canonical class and for $k = 1$, was first shown by Ding-Tian [30] with $A = 1/V(X)$ equal to the inverse of the volume of $-K_X$. This is the sharp constant in case X admits holomorphic vector fields. More precisely, they showed that any potential of a Kähler-Einstein metric on X optimizes the

corresponding Moser-Trudinger inequality (when dV is taken to depend on ω in a standard way). In case X has no holomorphic vector field the constant $A = 1/V(X)$ may be improved slightly as shown in the coercivity estimate of Phong-Song-Sturm-Weinkove [46] (confirming a previous conjecture of Tian).

In the case of a general Fano manifold Ding [29] obtained, using the Green function estimate of Bando-Mabuchi, a Moser-Trudinger inequality for all u in $\mathcal{H}_0(X, \omega)$ with a uniform positive lower bound ϵ on the Ricci curvature of the corresponding Kähler metric ω_u (for $k = 1$). The case of Theorem 1.1 for the anti-canonical class (but possibly no Kähler-Einstein metric) and with $k = 1$ was recently shown in [8], building on [14]. The approach in [8, 12] will be further developed in the present paper.

The setting of domains. A quasi-sharp version of the Brezis-Merle type inequality 1.10 was recently shown by Åhag-Cegrell-Kołodziej-Pham-Zeriahi [1]. More precisely it was shown that the inequality holds when raising the bracket in 1.10 to the power n . However the relation to the Moser-Trudinger inequality does not seem to have been noted before and we use it, among other things, to slightly improve the inequality in [1] with one power. The proof uses the “thermodynamical formalism” recently introduced in [9] (in the Kähler setting) and shows that the Moser-Trudinger inequality is equivalent to yet another inequality, coinciding with the classical *logarithmic Hardy-Sobolev inequality* when $n = 1$. As explained in [9] the corresponding inequality in the Kähler setting amounts to the boundedness from below of Mabuchi’s K-energy functional.

Towards the end of the writing of the present paper the preprint [21] appeared where the existence of solutions in Theorem 1.7 and Moser-Trudinger inequalities is proved under the stronger assumption that $a^{1/n} < n$.

Let us finally point out that Demailly [27] originally showed that a weaker version of inequality 1.10) is equivalent to a local algebra inequality previously obtained in [25] in the context of the study of birational rigidity of Fano manifolds. This latter inequality says that

$$(1.13) \quad \text{lc}(\mathcal{I}) \geq n/(\text{e}(\mathcal{I}))^{1/n},$$

where $\text{lc}(\mathcal{I})$ is the log canonical threshold of an ideal \mathcal{I} of germs of holomorphic functions and $\text{e}(\mathcal{I})$ is its Samuel multiplicity.

1.3. Outline of the proof of Theorems 1.1, 1.2. As is well-known a Kähler form ω is integral precisely when it can be realized as the (normalized) curvature form of a metric h on an ample line bundle $L \rightarrow X$. Abusing notation slightly this means that

$$\omega = dd^c \phi_0$$

where $h = e^{-\phi_0}$ is the expression of the metric h wrt a local holomorphic frame. Hence, ω_u is the curvature form of the metric on L with weight $\phi := \phi_0 + u$. The proof of Theorem 1.1 follows the same outline as the proof of the Moser-Trudinger inequality in [8, 14] concerning the case when $L = -K_X$ and ϕ_0 is the weight of a Kähler-Einstein metric - with some important modifications. The proof in [8, 14] is based on consideration of the functional

$$\mathcal{G}(\phi) := \log \int_X e^{-\phi} + \frac{1}{V} \mathcal{E}(\phi, \phi_0),$$

where we have used that $e^{-\phi}$ defines a global volume form on X (since $L = K_X$) and where $\mathcal{E}(\phi, \phi_0) := \mathcal{E}_\omega(\phi - \phi_0)$. The Moser-Trudinger inequality says that \mathcal{G} is negative on the space $\mathcal{H}(-K_X)$ of positively curved metrics on $-K_X$. But \mathcal{G} is geodesically concave on the space $\mathcal{H}(-K_X)$ equipped with the Mabuchi metric (see the next section) and the Kähler-Einstein condition says that ϕ_0 is

a critical point of \mathcal{G} . Moreover, by definition \mathcal{G} vanishes at $\phi = \phi_0$ and that ends the proof.

At first glance, not much of this argument works in our situation of a general line bundle $L \rightarrow X$. The functional

$$\phi \mapsto \log \int_X e^{-(\phi - \phi_0)} dV$$

has no obvious concavity properties and we have in general nothing that corresponds to the Kähler-Einstein condition. To handle the lack of concavity, we use a different functional, defined for each point x in X :

$$\phi \mapsto \log(K_{\phi_0}(x)/K_{\phi}(x)),$$

where K_{ϕ} is the restriction to the diagonal of the Bergman kernel for the space of global sections $H^0(X, L + K_X)$ of the adjoint line bundle $L + K_X$, which is known to be concave by the results in [12, 13]. It then turns out that we can replace the Kähler-Einstein condition by a standard estimate for the Bergman kernel in terms of the volume form; see [8] where a similar argument was used. The remaining problem is then to get from an estimate of the Bergman kernel to an estimate of the metric on L itself. On a compact manifold, this can be done using the basic formula

$$\int_X K_{\phi}(x) e^{-\phi} = N$$

where N is the dimension of $H^0(X, K_X + L)$. The growth rate in k in the inequality of the theorem is a consequence of a the Bergman kernel estimate, using that $k\phi$ is the weight of a metric on the k th tensor power of L , written as kL in our additive notation.

As for Theorem 1.2 it is proved by noting that the Bergman kernel estimate can be made to be uniform over all Fano manifolds of the same dimension by picking a reference metric ϕ_0 whose curvature form has a universal lower bound on its Ricci curvature.

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1.4. Notation and preliminaries. Here we will briefly recall the notions of (quasi-) psh functions and finite energy spaces in setting of compact manifolds X and domains Ω . In practice, it will, by approximation, be enough to prove the inequalities we will be interested in for smooth (or bounded) functions. However, the finite energy spaces play an important role in the variational approach used in section 7.

The setting of a compact manifold X . Let (X, ω) be a compact complex manifold and ω a smooth real closed $(1, 1)$ -form on X such that $\omega \geq 0$. We will mainly be concerned with the case when $\omega > 0$, i.e. when (X, ω) is a Kähler manifold. Denote by $PSH(X, \omega)$ be the space of all ω -psh functions u on X , i.e. $u \in L_1(X)$ and u is upper-semicontinuous (usc) and

$$\omega_u := \omega + \frac{i}{2\pi} \partial \bar{\partial} u := \omega + dd^c u \geq 0,$$

in the sense of currents (the normalizations are made so that $dd^c \log |z|^2 = 1$ when $n = 1$). We will write $\mathcal{H}(X, \omega)$ for the interior of $PSH(X, \omega) \cap C^\infty(X)$ (called the space of Kähler potentials when $\omega > 0$) and $\mathcal{H}_0(X, \omega)$ for its subspace

defined by the normalization $\sup_X u = 0$. We will also use the (non-standard) notion $\mathcal{H}(X, \omega)_b := PSH(X, \omega) \cap L^\infty(X)$ for the bounded functions in $PSH(X, \omega)$. By the local theory of Bedford-Taylor the Monge-Ampere operator

$$MA(u) := \omega_u^n / n!$$

is well-defined on $\mathcal{H}(X, \omega)_b$ and continuous under sequences decreasing to elements in $\mathcal{H}(X, \omega)_b$ as are all powers ω_u^p . In particular, the functional \mathcal{E}_ω (formula 1.3) is well-defined and continuous in the previous sense. Following [16, 10] \mathcal{E}_ω may be extended to all of $PSH(X, \omega)$ by setting

$$\mathcal{E}_\omega(u) := \inf_{v \in \mathcal{H}(X, \omega)_b, v \geq u} \mathcal{E}_\omega(v) \in [-\infty, \infty[$$

Now the space $\mathcal{E}^1(X, \omega)$ of all ω -psh functions of *finite energy* may be defined as the set of all u such that $\mathcal{E}_\omega(u) > -\infty$. As explained in [16, 10] it coincides with the space with the same name introduced in [35].

Metrics/weights on a line bundle vs. ω -psh functions. In the integral case, i.e. when $[\omega] = c_1(L)$ for a holomorphic line bundle $L \rightarrow X$, the space $PSH(X, \omega)$ may be identified with the space of (singular) Hermitian metrics on L with positive curvature current. More precisely, let s be a trivializing local holomorphic section of L , i.e. s is non-vanishing on a given open set U in X . First we identify an Hermitian metric $h_0 = \|\cdot\|$ on L with its *weight* ϕ , which is locally defined by the relation

$$\|s\|^2 = e^{-\phi_0}$$

The (normalized) curvature ω of the metric is the globally well $(1, 1)$ -current defined by the following local expression:

$$\omega = dd^c \phi_0$$

The identification with $PSH(X, \omega)$ referred to above is now obtained by fixing ϕ_0 and letting $\phi \mapsto u := \phi - \phi_0$ so that $dd^c \phi = \omega_u$. We will denote by \mathcal{H}_L the space of all semi-positively curved metrics/weights on L .

The setting of a domain Ω in \mathbb{C}^n . Let Ω be a bounded domain \mathbb{C}^n (in this setting $\omega = 0$) which is *hyperconvex*, i.e. it admits a negative continuous psh exhaustion function (for example a pseudoconvex domain with Lipschitz continuous boundary). The main reason that we will consider general hyperconvex domains (with possible non-smooth boundary) is that this property is preserved under Cartesian products. When Ω has smooth boundary we let $\mathcal{H}_0(\Omega)$ be the subspace of all smooth psh functions on $\bar{\Omega}$ such that $u = 0$ on $\partial\Omega$. Following [20, 1] (see also [6] for a comparison with the Kähler setting) it will also be convenient to use two singular versions of $\mathcal{H}_0(\Omega)$, namely $\mathcal{F}(\Omega)$ and $\mathcal{E}_1(\Omega)$, where the Monge-Ampère mass $\mathcal{M}(u)$ 1.11 and energy $\mathcal{E}_0(= \mathcal{E})$ 1.7 are well-defined and finite, respectively. More precisely, let first $\mathcal{H}_0(\Omega)_b$ be the space all u in $PSH(\Omega) \cap L^\infty(\Omega)$ such that $\mathcal{M}(u) < \infty$ and such that $\lim_{\zeta \rightarrow z} u(\zeta) = 0$ for any $z \in \partial\Omega$ (called the space of psh “test-functions” $\mathcal{E}_0(\Omega)$ in [20]). Now $\mathcal{F}(\Omega)$ is defined as the space of all u such that there exists $u_j \in \mathcal{H}_0(\Omega)_b$ decreasing to u with $\mathcal{M}(u_j) \leq C$. The Monge-Ampère operator extends to $\mathcal{E}_0(\Omega)$ in is continuous under decreasing limits. As for the space $\mathcal{E}_1(\Omega)$ it is defined in a similar manner, but by demanding that $-\mathcal{E}(u_j) \leq C$. There is also an alternative characterization of $\mathcal{F}(\Omega)$ as the set of all u in the “domain of definition of the Monge-Ampère operator” such that u has finite total Monge-Ampère mass and with smallest maximal plurisubharmonic majorant equal to zero (see

For the purpose of the present paper it will in practice be enough to know that if $u \in PSH(\Omega) \cap L^\infty(\Omega)$ such that $\lim_{\zeta \rightarrow z} u(\zeta) = 0$ for any $z \in \partial\Omega$, then $u \in \mathcal{F}(\Omega)$ if $\int_\Omega (dd^c u)^n < \infty$ and similarly $u \in \mathcal{E}^1(\Omega)$ if $\int_\Omega (-u)(dd^c u)^n < \infty$ (see [20, 1]).

It may also be convenient to recall (even if, strictly speaking, it will not be needed) the approximation result in [24] saying that *any* negative psh function u on a hyperconvex domain Ω can be written as decreasing limit of “smooth test functions”, i.e. psh functions u_j in $\mathcal{C}(\bar{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, vanishing on the boundary and with finite Monge-Ampère mass. As a consequence one may as well replace the space $\mathcal{H}_0(\Omega)_b$ in the previous definitions with the space of “smooth test functions” in the previous sense.

2. MOSER-TRUDINGER INEQUALITIES ON KÄHLER MANIFOLDS

Let X be an n -dimensional compact Kähler manifold and let L be a semi-positive line bundle over X and assume that L is big, i.e.

$$V = \int_X (dd^c \phi)^n / n! > 0$$

for any (and hence all) ϕ in $\mathcal{H}(L)$. We fix $\phi_0 \in \mathcal{H}(L)$ and let $\omega := dd^c \phi_0$.

2.1. Energy, geodesics and Bergman kernels (preliminaries). Given ϕ and ϕ_0 in $\mathcal{H}(L)$ we define (minus) the relative energy by

$$\mathcal{E}(\phi, \phi_0) = \frac{1}{(n+1)!} \int_X (\phi_0 - \phi) \sum_0^n (dd^c \phi_0)^k \wedge (dd^c \phi)^{n-k}$$

If $t \rightarrow \phi_t$ is a smooth curve in $\mathcal{H}(L)$ and

$$\dot{\phi}_t := \frac{d\phi_t}{dt}$$

then

$$\frac{d}{dt} \mathcal{E}(\phi_t, \phi_0) = \int_X \dot{\phi}_t (dd^c \phi_t)^n / n!.$$

This formula, together with the normalization $\mathcal{E}(\phi_0, \phi_0) = 0$ can also be used to define \mathcal{E} .

A basic property of \mathcal{E} is that it is linear along *geodesics* in $\mathcal{H}(L)$ and concave along subgeodesics defined wrt Mabuchi’s Riemannian metric on $\mathcal{H}(L)$. For technical reasons we will work with the following weaker notion of geodesics. Given two smooth metrics ϕ_0 and ϕ_1 the corresponding geodesic ϕ_t is defined as the following regularized envelope:

$$\phi_t := \Phi(z, t) := \sup_{\psi \in \mathcal{K}} \{\Psi(z, t)\}^*$$

where we have extended t to the strip $\mathcal{T} = [0, 1] + i\mathbb{R}$ in \mathbb{C} and \mathcal{K} is the set of all semi-positively curved metrics Ψ on the pull-back of L to $X \times \mathcal{T}$ such that $\psi_0 \leq \phi_0$ and $\psi_1 \leq \phi_1$. We will sometimes refer to a curve $\psi_t := \Psi(\cdot, t)$ above as a *subgeodesic*. When L is ample it was shown in [7] that Ψ is a continuous solution to the Dirichlet problem for the Monge-Ampère operator on $M := X \times \mathcal{T}$, i.e.

$$(dd^c \Phi)^{n+1} = 0$$

in the interior of M (in the usual sense of pluripotential theory) and on the boundary ∂M the metric Φ coincides with the $i\mathbb{R}$ invariant boundary data determined by ϕ_0 and ϕ_1 . However, we will only need some very modest regularity properties of Φ , namely that Φ is locally bounded and that $\Phi(t, \cdot) = \phi_t$ converges uniformly to the given boundary data as t approaches $\partial \mathcal{T}$. As shown by a simple barrier argument this is always the case as long as L is semi-positive (see [14]). Indeed,

$$(2.1) \quad \chi_t := \max\{\phi_0 - A\Re t, \phi_1 - A(1 - \Re t)\}$$

gives a candidate for the sup defining ϕ_t converging uniformly towards the right boundary values. Hence so does ϕ_t . Also note that, by imposing S^1 -symmetry in the complex variable t we might as well replace \mathcal{T} with an annulus \mathcal{A} .

Lemma 2.1. *Let ϕ_t be a (weak) geodesic as above. Then $t \mapsto \mathcal{E}(\phi_t, \phi_0)$ is affine and continuous up to the boundary of $[0, 1]$. Moreover, if $\dot{\phi}_0$ denotes the right derivative of ϕ_t at $t = 0$ (which exists by convexity), then*

$$\frac{d}{dt}_{t=0^+} \mathcal{E}(\phi_t) \leq \int_{\mathcal{B}} \dot{\phi}_0 (dd^c \phi_0)^n / n!,$$

As pointed out above this is well-known in the case when ϕ_t is smooth and follows immediately from the formula

$$(2.2) \quad d_t d_t^c \mathcal{E}(\phi_t, \phi_0) = \int_X (dd^c \Phi)^{n+1}$$

The general case then follows by approximation (see [8]); see also Prop 3.4 for the corresponding properties in the setting of domains.

Any element ϕ in $\mathcal{H}(L)$ defines an L^2 metric on $H^0(X, K_X + L)$,

$$\|u\|_{\phi}^2 = i^{n^2} \int u \wedge \bar{u} e^{-\phi}.$$

The Bergman kernel for this L^2 -metric is denoted $K_{\phi}(x)$. It can be defined as in the introduction

$$K_{\phi}(x) = i^{n^2} \sum u_j(x) \wedge \bar{u}_j(x)$$

where u_j is an orthonormal basis for $H^0(X, K_X + L)$. Alternatively,

$$(2.3) \quad K_{\phi}(x) = \sup_{H^0(X, K_X + L)} \{|u(x)|^2; \|u\|_{\phi} \leq 1\}.$$

Here the expression $|u(x)|^2$ depends on the choice of a trivialization of L near x , but $\log K_{\phi}$ is invariantly defined as a metric on $K_X + L$. As a consequence, the quotient of two Bergman kernels

$$K_{\phi}(x)/K_{\phi_0}(x)$$

is a global function on X , smooth if the sections in $H^0(X, K_X + L)$ have no common zeros.

We will use a result from [12] saying that

$$t \rightarrow \log K_{\phi_t}(x)$$

is, for any x fixed, convex along (sub)geodesics ϕ_t .

The first result we will need is the following simple formula for the derivative of the Bergman kernel along a curve (see for example the appendix in [8]).

Lemma 2.2. *Let ϕ_t be a smooth curve in $\mathcal{H}(L)$. Then*

$$\frac{d}{dt} K_{\phi_t}(x) = \int_X \dot{\phi}_t |K_{\phi_t}(x, y)|^2 e^{-\phi_t}$$

where the off-diagonal Bergman kernel is

$$K_{\phi_t}(x, y) := \sum c_n u_j(x) \wedge \bar{u}_j(y)$$

for any orthonormal basis of $H^0(X, K_X + L)$.

2.2. Moser-Trudinger type inequalities. The next proposition is the crux of the proof of the Moser-Trudinger inequalities.

Proposition 2.3. *Let ϕ and ϕ_0 be two metrics in $\mathcal{H}(L)$, satisfying the normalizing condition*

$$\phi - \phi_0 \leq 0.$$

Assume that the Bergman kernel for ϕ_0 satisfies

$$(2.4) \quad K_{\phi_0} e^{-\phi_0} \leq C_1 (dd^c \phi_0)^n / n!$$

Then

$$\sup_X \log \frac{K_{\phi_0}}{K_{\phi}} \leq C_1 \mathcal{E}(\phi, \phi_0).$$

Proof. Join ϕ_0 and ϕ with a geodesic ϕ_t such that $\phi_1 = \phi$. By the previous lemma

$$-\frac{d}{dt} \Big|_{t=0} \log K_{\phi_t}(x) = \int_X -\dot{\phi}_0 \frac{|K_{\phi_0}(x, y)|^2}{K_{\phi_0}(x)} e^{-\phi_0}.$$

Since ϕ_t is a geodesic, ϕ_t is convex in t , so

$$\dot{\phi}_0 \leq \phi - \phi_0 \leq 0.$$

Hence, since by Cauchy's inequality

$$\begin{aligned} |K_{\phi_0}(x, y)|^2 &\leq K_{\phi_0}(x) K_{\phi_0}(y), \\ -\frac{d}{dt} \Big|_{t=0} \log K_{\phi_t}(x) &\leq \int_X -\dot{\phi}_0 K_{\phi_0}(y) e^{-\phi_0}, \end{aligned}$$

which in turn is dominated by

$$C_1 \int_X -\dot{\phi}_0 (dd^c \phi_0)^n / n! \leq C_1 \frac{d}{dt} \Big|_{t=0} \mathcal{E}(\phi_t, \phi_0)$$

by the definition of C_1 (formula 2.4) and Lemma 2.1 which also gives

$$\frac{d}{dt} \Big|_{t=0} \mathcal{E}(\phi_t, \phi_0) = \mathcal{E}(\phi, \phi_0).$$

Now we use that $f(t) : -\log K_{\phi_t}$ is concave. Therefore

$$f(1) - f(0) \leq f'(0)$$

which means that

$$\log K_{\phi_0} - \log K_{\phi} \leq f'(0) \leq C_1 \mathcal{E}(\phi, \phi_0)$$

which completes the proof. \square

Now it only remains to convert this estimate of the Bergman kernel to an estimate of the integral of $e^{-\phi}$. Here we use

$$(2.5) \quad \int_X K_{\phi} e^{-\phi} = N := \dim H^0(X, L + K_X)$$

Let C_1 and C_2 be constants satisfying

$$(2.6) \quad C_2 dV \leq K_{\phi_0} e^{-\phi_0} \leq C_1 (dd^c \phi_0)^n / n!$$

where dV is a fixed volume form on X . Note that $L + K_X$ is basepoint free precisely when C_2 can be taken to be strictly positive.

By the previous proposition and 2.6 we have for any x in X

$$(2.7) \quad K_{\phi} \geq K_{\phi_0} e^{-C_1 \mathcal{E}(\phi, \phi_0)} \geq C_2 e^{\phi_0} dV e^{-C_1 \mathcal{E}(\phi, \phi_0)},$$

so it follows that

$$\int_X e^{-(\phi - \phi_0)} dV \leq C_2^{-1} N e^{-C_1 \mathcal{E}(\phi, \phi_0)}.$$

We collect this in the next theorem which, as explained below, implies Theorem 1.1 in the introduction.

Theorem 2.4. *Let ϕ_0 be a semipositively curved metric on the line bundle L over the compact Kähler manifold X .*

Assume that the Bergman kernel for ϕ_0 satisfies 2.6. Then for any other semipositively curved metric on L , satisfying

$$\phi - \phi_0 \leq 0.$$

we have that

$$\log \int_X e^{-(\phi - \phi_0)} dV \leq \log(N/C_2) - C_1 \mathcal{E}(\phi, \phi_0).$$

We say (cf [13],[8]) that the metric ϕ_0 is *balanced in the adjoint sense* if there is a constant C such that

$$K_{\phi_0} e^{-\phi_0} = C (dd^c \phi_0)^n / n!.$$

When $dV := (dd^c \phi_0)^n / n!$ this amounts to saying that the constants C_1 and C_2 in 2.6 can be chosen equal, and integrating over X we see that in this case $C = N/V$. We thus immediately get the next corollary.

Corollary 2.5. *With assumptions as in Theorem 2.3, assume in addition that ϕ_0 is balanced in the adjoint sense. Then*

$$\log \int_X e^{-(\phi - \phi_0)} (dd^c \phi_0)^n / n! \leq \frac{N}{V} \mathcal{E}(\phi, \phi_0).$$

As an example of this, let us look at the case $L = -K_X$. Then $H^0(X, K_X + L) = \mathbb{C}$, i.e. $N = 1$, and

$$K_{\phi_0}(x) = 1 / \int_X e^{-\phi_0}.$$

Hence the condition that ϕ_0 be balanced in the adjoint sense means that

$$(dd^c \phi_0)^n / (Vn!) = \left(\int_X e^{-\phi_0} \right)^{-1} e^{-\phi_0}$$

which means that ϕ_0 is the potential of a Kähler-Einstein metric. Then the corollary becomes

$$\log \int_X e^{-\phi} \leq \log \int_X e^{-\phi_0} + \mathcal{E}(\phi, \phi_0)$$

since $N = 1$. This is the Moser-Trudinger inequality first proved in [30] (using a different method). Note that the assumption that $\phi \leq \phi_0$ is unnecessary here since both sides scale the same way if we subtract a constant from ϕ .

2.2.1. Proof of Theorem 1.1. In a similar vein we can consider asymptotic versions of Theorem 2.3, when we replace L by kL , with k a large integer. Then it follows from well-known Bergman kernel asymptotics due to Bouche and Tian (see [54] and references therein for various refinements) that for any fixed smooth and strictly positively curved ϕ_0

$$(2.8) \quad K_{k\phi_0} e^{-k\phi_0} = (dd^c k\phi_0)^n / n! (1 + O(k^{-1}))$$

Hence in 2.6 we can take both C_1 and C_2 equal to

$$1 + O(k^{-1})$$

Integrating 2.8 we also get the well known formula

$$N_k = V k^n + o(k^{n-1})$$

for the dimension of the space of global sections of $K_X + kL$. Altogether this finishes the proof of Theorem 1.1.

2.3. Uniformity over all Fanos (proof of Theorem 1.2). We start with the following essentially well-known lemma which is proved using Moser iteration (see Thm. 7 in [43] which is stated for the equality case in 2.9, but the proof in general is the same)

Lemma 2.6. *Let (X, g) be a Riemannian manifold of real dimension $2n > 2$ and let a_g and b_g be constants such that the following Sobolev inequality holds for any function F on X such that F and its gradient are in L^2 :*

$$\left(\int_X |F|^{2\sigma} dV_g \right)^{1/\sigma} \leq \left(a_g \int_X |\nabla_g F|^2 dV_g + b_g \int_X |F|^2 dV_g \right), \quad \sigma = n/(n-1)$$

For any positive function H such that $\Delta_g H \geq -\lambda H$ there is a constant C_g only depending on a_g and b_g such that

$$(2.9) \quad \|H\|_{L^\infty(X)} \leq C_g \lambda^n \|H\|_{L^1(X, g)}$$

Let us now assume that $L \rightarrow X$ is a an ample line bundle with a fixed positively curved weight ϕ_0 such that the Kähler form $\omega_0 := dd^c \phi_0$ has a lower bound δ on its Ricci curvature:

$$(2.10) \quad \text{Ric } \omega_0 \geq \delta \omega_0$$

Then we claim that there is a constant C_δ only depending on δ such that the Bergman kernel $K_{k\phi_0}(x)$ of the space $H^0(kL + K_X)$ has the following point-wise upper bound:

$$(2.11) \quad K_{k\phi_0} \leq C_\delta k^n (dd^c \phi_0)^n / n!$$

To see this let g be the Riemannian metric on X corresponding to ω_0 . By [37] the corresponding constants a_g and b_g only depend on the lower bound δ of the Ricci curvature of g the lower bound on the volume V_g of g :

$$a_g := \frac{2n-1}{n(n-1)\delta V^{1/n}}, \quad b_g = \frac{1}{V^{1/n}}$$

Let now f_k be an element in $H^0(kL + K_X)$ and write

$$H := |f_k|^2 e^{-k\phi_0} / ((dd^c \phi_0)^n / n!)$$

Then it follows immediately from the definition of Ricci curvature and the fact that $\log |f_k|^2$ is locally psh that

$$dd^c \log H \geq -k\omega_0 - \delta\omega_0$$

and hence $dd^c H \geq -(k - \delta)H\omega_0$. Applying the previous Lemma to H with $\lambda := n(k - \delta)$ now gives

$$|f_k|^2 e^{-k\phi_0} \leq C_\delta k^n \frac{(dd^c \phi_0)^n}{n!} \int_X |f_k|^2 e^{-k\phi_0}.$$

By the extremal definition of $K_{k\phi_0}$ this finally proves the inequality 2.11.

Let us now assume that X is a Fano manifold and take $L := -K_X$ so that $V := c_1(-K_X)^n / n!$. As shown by Tian-Yau [51] one may always choose $\omega := \omega_0 \in c_1(-K_X)$ so that $1/\delta$ in 2.10 only depends on an upper bound on V (since changing ϕ_0 only changes the additive constant B_k we are allowed to choose ϕ_0 and dV). As later shown in [19, 41] the volume V of an n -dimensional Fano has a universal bound $V \leq c_n$ and hence ϕ_0 may be chosen so that the Bergman kernel estimate 2.11 holds with a constant C_δ only depending on the dimension n . The proof of Theorem 1.2 is now concluded by invoking Theorem 2.4.

Remark 2.7. One may also ask whether there is universal *lower* bound on $\inf_X (K_{k\phi} e^{-k\phi} / (dd^c \phi)^n)$ in terms of a positive lower bound δ of the Ricci curvature of $dd^c \phi$ and the dimension n of the Fano manifold? If one instead considers the Bergman kernel $\tilde{K}_{k\phi}$ defined wrt the L^2 -norm $\int_X |f|^2 e^{-k\phi} (dd^c \phi)^n$ on

$H^0(X, kK)$ then a lower bound for $\tilde{K}_{k\phi} e^{-k\phi}$ was obtained by Tian [49] when $n = 2$ (for all $\phi = \phi_t$ appearing in Aubin's continuity path) and it forms a crucial role in the proof in [49] of the Calabi conjecture on Fano (Del Pezzo) surfaces.

2.4. Vanishing Lelong numbers (proof of Cor 1.3). Let now ω be a semi-positive form with positive volume and L a semi-positive and big line bundle (i.e. $V > 0$) with $c_1(L) = [\omega]$. Take as before $\phi_0 \in \mathcal{H}_L$ with curvature form $\omega \geq 0$. In case $kL + K_X$ is semi-ample for all $k \geq k_0$ we have, by definition, that $K_{k\phi_0} > 0$ on all of X and hence $C_2 > 0$ so that Cor 1.3 follows immediately from Theorem 2.4. In the general case we proceed as follows. Given ϕ a locally bounded weight with $dd^c\phi \geq 0$ 2.7 gives

$$(2.12) \quad e^{C_1 C} \int_X K_{k\phi_0} e^{-k\phi} \leq \int_X K_{k\phi} e^{-k\phi} = N_k$$

Next, we claim that there is a holomorphic section s_E of a holomorphic line bundle $E \rightarrow X$ with a smooth weight ϕ_E such that

$$(2.13) \quad |s_E|^2 e^{-\phi_E} / C' \leq K_{k\phi_0} e^{-k\phi_0}$$

for some constant C' . This is an essentially well-known consequence of the Ohsawa-Takegoshi-Manivel extension theorem proved as follows. Since L is big we may, by Kodaira's lemma, decompose $k_0 L = A + E$ for A ample (positive) and E effective, i.e. A admits a positively curved weight ϕ_A and E admits a holomorphic section s_E . Then $\psi := \phi_A + \log |s_E|^2$ defines a positively curved weight on L such that its curvature $dd^c\psi \geq dd^c\phi_A := \omega_A$ is a Kähler current. We also fix a smooth weight ϕ_{K_X} on K_X . By the Ohsawa-Takegoshi-Manivel extension theorem k_0 may be chosen sufficiently large so that for any fixed point $x \in X$ there is a holomorphic section $f_k \in H^0(kL + K_X)$ such that

$$|f_k|^2(x) e^{-((k-k_0)\phi_0 + \psi + \phi_{K_X})} = 1, \quad \int_X |f_k|^2 e^{-((k-k_0)\phi_0 + \psi)} \leq C''$$

for an absolute constant C'' . Since, we may after, subtracting a constant, assume that $\psi \leq \phi_0$ we can replace the ψ in the inequality above with ϕ_0 and hence 2.13 follows from the extremal property of the Bergman kernel.

Combining 2.12 and 2.6 now gives the existence of a constant C such that

$$(2.14) \quad \int_X |s_E|^2 e^{\phi_0 - \phi_E} e^{-k\phi} \leq C N_k e^{-C\mathcal{E}(\phi, \phi_0)}$$

for any fixed k and ϕ as above with $\sup(\phi - \phi_0) = 0$. Hence, if ϕ is a weight of finite energy it follows from the definition that the integral in the lhs above is finite. But then it follows from a simple local argument that a local representative of ϕ cannot have any Lelong numbers at given point x in X . Indeed, if ϕ had Lelong number $l > 0$ at $z = 0$ in local coordinates z on a small ball \mathcal{B} , then $\phi \leq l \log |z|^2 + C$. If we now blow-up the point $z = 0$ and denote by s the section cutting out the exceptional divisor E_0 on the blow-up \mathcal{B}_0 of \mathcal{B} we get

$$\int_{\mathcal{B}_0} |s|^{2m} |s|^{-2(lk - (n-1))} < \infty$$

where m is the order of vanishing of s_E along E_0 . Hence, taking k sufficiently large (i.e. so that $kl > m + n - 1$) finally yields the desired contradiction, showing that ϕ has no Lelong numbers, which by Skoda's result is equivalent to $e^{-k\phi} \in L_{loc}^1$ for any $k \geq 0$ (see for example [28]). Finally, since $\{\mathcal{E} \geq -C\} \cap \{\sup_X = 0\}$ is a compact set in $PSH(X, \omega)$ where along Lelong numbers vanish identically the last statement of the corollary follows from either from the uniform version of Skoda's theorem in [55] (just as in the proof of Lemma 6.4

in [10]) or alternatively the semi-continuity of complex integrability exponents established in [28].

3. MOSER-TRUDINGER INEQUALITY IN THE BALL UNDER S^1 -INVARIANCE

In this section we will look at estimates for integrals of $e^{-\phi}$, where ϕ is plurisubharmonic in a domain in \mathbb{C}^n . For simplicity we will treat only the ball \mathcal{B} . As in the previous section we let $K_\phi(x)$ be the Bergman kernel at the diagonal for the plurisubharmonic weight function ϕ . It follows from the results in [12] that $\log K_{\phi_t}(x)$ is convex in t if $t \rightarrow \phi_t$ is a geodesic in the space of plurisubharmonic functions in \mathbb{B} (see below).

We say that a function f is S^1 -invariant if $f(e^{i\theta}z) = f(z)$. (Here $e^{i\theta}$ acts diagonally so that $e^{i\theta}(z_1, \dots, z_n) := (e^{i\theta}z_1, \dots, e^{i\theta}z_n)$). We also say that a domain is S^1 -invariant if $e^{i\theta}z$ lies in the domain if z does.

3.1. Bergman kernels and plurisubharmonic variations.

Proposition 3.1. *Assume ϕ is plurisubharmonic in an S^1 invariant (connected) domain that contains the origin and that ϕ is also S^1 -invariant. Then*

$$K_\phi(0, \zeta) = 1 / \int e^{-\phi}$$

for all ζ in the ball.

Proof. By definition, $K_\phi(0, \zeta)$ is antiholomorphic in ζ and by uniqueness of Bergman kernels it must also be S^1 -invariant. Hence it is a constant, and since

$$\int 1 K_\phi(0, \cdot) e^{-\phi} = 1$$

the proposition follows. \square

The next proposition then follows immediately from the plurisubharmonic variation of Bergman kernels (cf [12]).

Proposition 3.2. *Let ϕ_t be a subgeodesic of S^1 -invariant plurisubharmonic functions in the disk. Then*

$$t \mapsto \log \left(\int e^{-\phi_t} \right)$$

is concave.

3.2. Energy and geodesics. In this section we will adapt the results about geodesics and energy in the compact Kähler setting to the setting of domains. In principle all the previous properties go through in this latter setting. The main technical difference is that one has to be a bit careful when performing integration by parts, due to the presence of the boundary. For this reason it will be convenient to work in the singular setting of the finite energy class $\mathcal{E}(\Omega)$ (compare section 1.4).

In a domain Ω we have a variant of the energy \mathcal{E} , which in case ϕ_0 and ϕ are smooth is defined by

$$\mathcal{E}(\phi, \phi_0) = \frac{1}{(n+1)!} \int_{\Omega} (\phi - \phi_0) \sum_0^n (dd^c \phi_0)^k \wedge (dd^c \phi)^{n-k}$$

and when $\phi = \phi_0 = 0$ on Ω integration by parts show that $\mathcal{E}(\phi, \phi_0) = \mathcal{E}(\phi) - \mathcal{E}(\phi_0)$ (compare the lemma below), where

$$\mathcal{E}(\phi) := \mathcal{E}_0(\phi) := \mathcal{E}(\phi, 0)$$

so that

$$\mathcal{E}(\phi) = \frac{1}{(n+1)!} \int_{\mathcal{B}} (-\phi) (dd^c \phi)^n.$$

Moreover, integration by parts also give

$$\frac{d}{dt}\mathcal{E}(\phi_t, \phi_0) = \int_{\mathcal{B}} \dot{\phi}_t (dd^c \phi_t)^n / n!$$

We will need the following generalization:

Lemma 3.3. *Let ϕ and ψ be in $\mathcal{E}^1(\Omega)$. Then*

$$\frac{d}{dt}_{t=0^+} \mathcal{E}(\phi + t(\psi - \phi)) = \int_{\mathcal{B}} (\psi - \phi)(dd^c \phi)^n / n!$$

Moreover, the following cocycle relation holds $\mathcal{E}(\phi) - \mathcal{E}(\psi) = \mathcal{E}(\phi, \psi)$.

Proof. Assume first that ϕ and ψ are in $\mathcal{H}_0(\Omega)_b$. In this class one may integrate by parts just as in the smooth case (using the assumption on finite Monge-Ampère mass; see [20] and references therein) and hence expanding $\mathcal{E}(\phi + t(\psi - \phi))$ and integrating by parts gives

$$\mathcal{E}(\phi + t(\psi - \phi)) = t \int_{\mathcal{B}} (\psi - \phi)(dd^c \phi)^n + O(t^2)I$$

where I is a sum of terms of the form $\int (\psi - \phi)(dd^c \phi)^{n-j}(dd^c \psi)^j$ which are finite since ϕ and ψ are in $\mathcal{H}(\Omega)_b$. This finishes the proof in the case of the class $\mathcal{H}_0(\Omega)_b$. Finally, given ϕ and ψ in $\mathcal{E}^1(\Omega)$ we take sequences ϕ_j and ψ_k in $\mathcal{H}(\Omega)_b$, decreasing to ϕ and ψ respectively. By the previous case we have

$$\mathcal{E}(\phi_j + t(\psi_k - \phi_j)) = \int_0^t \int_{\Omega} (\psi_k - \phi_j)(dd^c(\phi_j + s(\psi_k - \phi_j)))^n ds =: \int_0^t g_{k,j}(s) ds$$

By well-known continuity properties [20] and the finite energy assumptions letting first j and then k tend to infinity shows that the previous formula holds with ϕ_j and ψ_k replaced with ϕ and ψ , respectively. Moreover, for the same reason the corresponding density $g(s)$ is continuous wrt s and that ends the proof of the derivative formula in the general case. Finally, the previous formula implies the cocycle relation by integrating along the line $t \mapsto \phi + t(\psi - \phi)$ (note that by a well-known Cauchy-Schwartz type estimate all terms in $\mathcal{E}(\phi, \psi)$ are finite). \square

Next we turn to the definition of geodesic segments in the setting of domains. Given, say ϕ_0 and ϕ_1 on Ω which are psh and smooth up to the boundary, where they vanish, the corresponding geodesic ϕ_t is defined by replacing the space $\mathcal{H}(L)_b$ with the space of all bounded psh functions tending to zero at the boundary. In other words, a geodesic is defined as the following regularized envelope, where $M := \Omega \times \mathcal{A}$ (with \mathcal{A} denoting an annulus):

$$\phi_t := \Phi(z, t) := \sup_{\psi \in \mathcal{K}} \{\Psi(z, t)\}^*$$

where \mathcal{K} is the set of all psh functions $\Psi \in PSH \cap L^\infty(M)$ such that $\Psi^* \leq f$ on ∂M , where f is the function on ∂M defined as follows: decomposing $\partial M := B_1 \cup B_2 := \partial\Omega \times \mathcal{A} \cup \partial\mathcal{A}$ we let $f = 0$ on B_1 and $f = \phi_i$ for $i = 1, 2$ on the two different components of B_2 . In particular, if the ϕ_0 and ϕ_1 are continuous on $\bar{\Omega}$ then so is the boundary data f . Just as in the setting of compact Kähler manifolds we may as well, by symmetry, replace the bounded domain \mathcal{A} with a strip so that, for t real, ϕ_t gets identified with a function on $\Omega \times [0, 1]$. In this latter notation there is a similar construction of a barrier χ_t as in the compact case, namely

$$(3.1) \quad \chi_t := \max\{\phi_0 - A\Re t, \phi_1 - A(1 - \Re t), A\rho\}$$

where ρ is a psh exhaustion function of Ω (e.g. $\rho = |z|^2 - 1$ in the ball case). It hence determines an extension F of f such that $F \in \mathcal{C}^0(\bar{M}) \cap PSH(M)$ and hence Φ is bounded on M and converges uniformly towards the right boundary

vales. In fact, given the extension F above it follows from a theorem in [15] (since M is hyperconvex) that $\Phi \in \mathcal{C}^0(\bar{M}) \cap PSH(M)$ with

$$(dd^c \Phi)^{n+1} = 0, \text{ in } M$$

(but strictly speaking we will not need the continuity, only the boundedness and the uniform boundary behavior as $t \rightarrow 0$ and $t \rightarrow 1$). In particular we obtain a continuous curve ϕ_t in the space $PSH \cap L^\infty(\Omega)$.

Lemma 3.4. *Let ϕ_t be a geodesic segment as above.*

- *For any fixed t we have that $\phi_t \in \mathcal{E}^1(\Omega)$ and if $\dot{\phi}_0$ denotes the right derivative of ϕ_t at $t = 0$ (which exists by convexity), then*

$$\frac{d}{dt}_{t=0^+} \mathcal{E}(\phi_t) \leq \int_{\mathcal{B}} \dot{\phi}_0 (dd^c \phi_0)^n / n!,$$

- *$t \mapsto \mathcal{E}(\phi_t)$ is affine and continuous on $[0, 1]$.*

Proof. As explained above $\chi_t \leq \phi_t \leq 0$ where χ_t is a maximum of functions in $\mathcal{E}^1(\Omega)$ and hence χ_t is also in the space $\mathcal{E}^1(\Omega)$ [20]. By Lemma 3.3 the functional \mathcal{E} is increasing on $\mathcal{E}^1(\Omega)$ (since its differential is a positive measure) and hence $-\infty < \mathcal{E}(\chi_t) \leq \mathcal{E}(\phi_t^j) \leq 0$ for any sequence ϕ_t^j in $\mathcal{H}_0(\Omega)_b$ decreasing to ϕ , which proves the first claim. Next, we recall that \mathcal{E} is concave on $\mathcal{E}^1(\Omega)$ (wrt the usual affine structure) which for example follows from the formula for $d_t d_t^c \mathcal{E}(\phi_t)$ discussed below. In particular,

$$\frac{1}{t}(\mathcal{E}(\phi_t) - \mathcal{E}(\phi_0)) \leq \frac{1}{t} \int_{\Omega} (\phi_t - \phi_0) (dd^c \phi_0)^n / n!,$$

so that letting $t \rightarrow 0^+$ proves the first point. As for the last point integration by parts show that the formula 2.2 for $d_t d_t^c \mathcal{E}(\phi_t)$ is still valid in the smooth case. However, as we will need the formula in a singular setting we instead refer to the result proved in [1] which implies that if $\Phi \in \mathcal{F}(\otimes \times \mathcal{A})$ whose slices ϕ_t are in $\mathcal{E}^1(\Omega)$ then the analogue of formula 2.2 holds (i.e. for $X = \Omega$) in the sense of currents. Finally, since $\phi_t \rightarrow \phi_0$ uniformly as $t \rightarrow 0$ we have that $\mathcal{E}(\phi_t) \rightarrow \mathcal{E}(\phi_0)$ as $t \rightarrow 0$ [20] and similarly for $t \rightarrow 1$ and that ends the proof. \square

3.3. The case of the ball and the proof of Theorem 1.4. We now take Ω to be the unit-ball and make a special choice of reference function ϕ_0 as

$$\phi_0 = (n+1)[\log(1 + |z|^2) - \log 2].$$

This is a potential of the Fubini-Study metric on \mathbb{P}^n and satisfies the Kähler-Einstein equation

$$(3.2) \quad (dd^c \phi_0)^n / n! = a_n e^{-\phi_0}$$

We first prove an estimate for the Bergman kernel at the origin. By Prop 3.1 this amounts to an estimate of the integral of $e^{-\phi}$ in the S^1 -invariant case, but we prefer to argue first in the general case, since we feel the estimate for Bergman kernels has independent interest.

Proposition 3.5. *Let ϕ be a smooth plurisubharmonic function in the ball that vanishes on the boundary. Then*

$$-\log K_\phi(0) \leq \log \int_{\mathcal{B}} e^{-\phi_0} - b_n \mathcal{E}(\phi, \phi_0)$$

where $b_n = (a_n \int e^{-\phi_0})^{-1}$.

Proof. As in the compact setting the proof uses geodesics. We connect ϕ and ϕ_0 by a geodesic ϕ_t such that $\phi_1 = \phi$. Then

$$g(t) := \log K_{\phi_t} - b_n \mathcal{E}(\phi_t, \phi)$$

is a convex function of t . We claim that $g'(0) \geq 0$. For this we use the same formula as before for the derivative of the Bergman kernel

$$\frac{d}{dt}K_{\phi_t}(x) = \int_{\mathcal{B}} \dot{\phi}_t |K_{\phi_t}(x, y)|^2 e^{-\phi_t}.$$

Take $x = 0$ and $t = 0$. Then, since ϕ_0 is S^1 -symmetric

$$K_{\phi_0}(0, y) = 1 / \int_{\mathcal{B}} e^{-\phi_0}$$

by proposition 3.1. Therefore

$$\frac{d}{dt}|_{t=0} \log K_{\phi_t}(0) = \int_{\mathcal{B}} \dot{\phi}_0 e^{-\phi_0} / \int_{\mathcal{B}} e^{-\phi_0}.$$

Combining this with the Kähler-Einstein condition 3.2 we get, also using Lemma 3.4, that

$$\frac{d}{dt}|_{t=0} \log K_{\phi_t}(0) = b_n \int_X \dot{\phi}_0 (dd^c \phi_0)^n / n! \geq b_n \frac{d}{dt}|_{t=0} \mathcal{E}(\phi_t, \phi).$$

so $g'(0) \geq 0$ as claimed. Since g is moreover convex we get $g(1) \geq g(0)$ or explicitly

$$\log K_{\phi}(0) - b_n \mathcal{E}(\phi, \phi_0) \geq \log K_{\phi_0}(0).$$

Invoking 3.1 again the proposition follows. \square

From here we can not continue as in the compact case since we have no counterpart of 2.5. It seems plausible to conjecture that for any compact K in the ball

$$\int_K K_{\phi}(z, z) e^{-\phi(z)} \leq C(K, \phi)$$

where the constant depends only on K and, say,

$$\int_{\mathbb{B}} (dd^c(\phi + |z|^2))^n.$$

If this were true we could follow a route similar to what we did in the case of a compact manifold and obtain sharp estimates for

$$\int_K e^{-\phi}$$

for functions that are not necessarily S^1 -invariant. The most one could hope for in this direction would be

$$\int_{\mathbb{B}} (1 - |z|^2)^{n+1} K_{\phi}(z, z) e^{-\phi(z)} \leq C(\phi)$$

with the same dependence on ϕ . We do not know if either of these estimates hold.

Instead we now introduce the additional assumption that ϕ be S^1 -invariant. We then get, by Proposition 3.1, that

$$\log \int_{\mathcal{B}} e^{-\phi} \leq \log \int_{\mathcal{B}} e^{-\phi_0} - b_n \mathcal{E}(\phi, \phi_0)$$

if ϕ is any smooth plurisubharmonic function in the ball, vanishing on the boundary and S^1 -invariant.

As it stands the constant here is not optimal. An easy way to improve it is to replace our 'reference' ϕ_0 by

$$(3.3) \quad \phi_0^\epsilon := (n+1)[\log(\epsilon^2 + |z|^2) - \log(\epsilon^2 + 1)].$$

This amounts to replacing the unit ball by a larger ball of radius $1/\epsilon$ which brings us closer and closer to all of \mathbb{P}^n , where the same argument is known to give an optimal constant. Then

$$(dd^c \phi_\epsilon)^n / n! = a_n(\epsilon) e^{-\phi_\epsilon},$$

and as before we let

$$b_n(\epsilon) = 1 / (a_n(\epsilon) \int e^{-\phi_\epsilon}).$$

By the Kähler-Einstein equation for ϕ_ϵ

$$b_n(\epsilon) = n! / \int (dd^c \phi_\epsilon)^n.$$

The integral here is easily computed using Stokes' theorem

$$\begin{aligned} \int_{|z|<1} (dd^c \phi_\epsilon)^n &= \int_{|z|=1} d^c \phi_\epsilon \wedge (dd^c \phi_\epsilon)^{n-1} = \\ &= (n+1)^n (1+\epsilon^2)^{-n} \int_{|z|=1} d^c |z|^2 \wedge (dd^c |z|^2)^{n-1} = (n+1)^n (1+\epsilon^2)^{-n} \int_{|z|<1} (dd^c |z|^2)^n = \\ &= (n+1)^n (1+\epsilon^2)^{-n} n! \pi^{-n} |\mathbb{B}_n| = (n+1)^n (1+\epsilon^2)^{-n}. \end{aligned}$$

Hence $b_n(\epsilon)$ is asymptotic to $n!/(n+1)^n$ as ϵ goes to zero (coinciding with the inverse of the volume of \mathbb{P}^n , as it must). We have

$$\mathcal{E}(\phi_\epsilon) = (n+1)^{-1} \int_{\mathcal{B}} \phi_\epsilon (dd^c \phi_\epsilon)^n / n!$$

which by the Kähler-Einstein equation equals

$$-a_n(\epsilon) \int_{\mathcal{B}} \log(\epsilon^2 + |z|^2) e^{-\phi_\epsilon}$$

plus a quantity tending to zero with ϵ . Thus

$$b_n \mathcal{E}(\phi_\epsilon) = - \int_{\mathcal{B}} \log(\epsilon^2 + |z|^2) e^{-\phi_\epsilon} / \int_{\mathcal{B}} e^{-\phi_\epsilon}.$$

This is the integral of $-\log(\epsilon^2 + |z|^2)$ against a sequence of measures that tend to a Dirac unit mass at the origin, and it is easily seen to be asymptotic to a constant plus $-\log \epsilon^2$. On the other hand

$$-\log \int_{\mathcal{B}} e^{-\phi_\epsilon}$$

is also asymptotic to $-\log \epsilon^2$ plus a constant. All in all this proves Theorem 1.4 stated in the introduction.

Notice that there seems to be no extremal function for the inequality. For any nonzero ϵ , ϕ_0^ϵ is an extremal by construction, but these functions tend to $(n+1) \log |z|^2$, which has infinite energy.

We do not know if 1.4 holds without our assumption of S^1 -symmetry except for $n=1$, see [45] where a symmetrization argument can be used. Our methods also have bearings on symmetrization properties in the present higher dimensional setting of domains in \mathbb{C}^n and we hope to come back to this point in the future. In section 4 we shall use a different argument to prove the inequality 'modulo ϵ ' without assuming S^1 -invariance.

4. MOSER-TRUDINGER AND BREZIS-MERLE TYPE INEQUALITIES ON DOMAINS IN \mathbb{C}^n

Let Ω be a bounded hyperconvex domain in \mathbb{C}^n . We may then set the reference form ω_0 to be the zero-form: $\omega_0 = 0$ and use the notation

$$\mathcal{E}(u) := \mathcal{E}_{\omega_0}(u) = \frac{1}{(n+1)!} \int_{\Omega} u (dd^c u)^n$$

It will also be convenient to write

$$\mathcal{M}(u) := \int_{\Omega} (dd^c u)^n$$

We will say that the *sharp Moser-Trudinger (M-T) inequality* holds for the domain Ω if there is a constant C such that

$$(M-T) \quad \log \int_{\Omega} e^{-u} dV \leq -\frac{n!}{(n+1)^n} \mathcal{E}(u) + C$$

for any $u \in \mathcal{E}_1(\Omega)$. Similarly, the *quasi-sharp M-T inequality* is said to hold on Ω if for any $\delta > 0$ the previous inequality holds when the factor $n+1$ in front of $\mathcal{E}(u)$ is replaced by $n+1-\delta$ and the constant C by $C - \log(\delta^{(n-1)})$.

The *sharp Brezis-Merle (B-M) inequality* is said to hold for the domain Ω if there is a constant A such that

$$(B-M) : \quad \int_{\Omega} e^{-u} dV \leq A \left(1 - \frac{1}{n^n} \mathcal{M}(u) \right)^{-1}$$

for any $u \in \mathcal{F}(\Omega)$ such that $\mathcal{M}(u) := \int_{\Omega} (dd^c u)^n < n^n$.

It will also be convenient to use the following equivalent formulations of the quasi-sharp Moser-Trudinger and Brezis-Merle inequalities:

$$(M-T') \quad \int_{\Omega} e^{-(n+1-\delta)u} dV \leq C \delta^{-(n-1)} e^{-(n+1-\delta)n! \mathcal{E}(u)}$$

for some positive constant C and (when $n > 1$): there is a positive constant A such that

$$(4.1) \quad (B-M') : \quad \int_{\Omega} e^{-(n-\delta)u} dV \leq A \delta^{-(n-1)}$$

for all $u \in \mathcal{F}(\Omega)$ such that $\mathcal{M}(u) = 1$

4.1. M-T in \mathbb{C}^n implies B-M in \mathbb{C}^{n+1} .

Proposition 4.1. *The (quasi-) sharp Moser-Trudinger inequality on $\Omega \subset \mathbb{C}^n$ implies the (quasi-) sharp Brezis-Merle inequality on $\Omega \times D \subset \mathbb{C}^{n+1}$. More generally, the (quasi-) sharp Moser-Trudinger inequality on the ball in \mathbb{C}^n implies the (quasi-) sharp Brezis-Merle inequality on any hyperconvex domain in \mathbb{C}^{n+1} .*

Proof. Let us start with the sharp case. Given $u \in \mathcal{F}(\Omega_z \times D_t)$ we let $v(t) := \mathcal{E}(u(t, \cdot))$ and to fix ideas we first assume that u is smooth on the closure of $\Omega \times D$. Applying the sharp M-T inequality to $u(t, \cdot)$ for t fixed and integrating over $t \in D$ gives

$$\int_D \left(\int_{\Omega} e^{-u(t,z)} dV(z) \right) dV(t) \leq \int_D \exp\left(-\frac{n!}{(n+1)^n} v(t)\right) dV(t),$$

By 2.2 the function $v(t)$ is a subharmonic function on D with $\int_D d_t d_t^c v = \int_{\Omega \times D} (dd^c u)^{n+1} / (n+1)!$. Hence applying the sharp B-M inequality on the disc D for $n = 1$ (which follows from Green's formula and Jensen's inequality [17] or alternatively from Polya's inequality [1]) and using that $\frac{n!}{(n+1)^n} \frac{1}{(n+1)!} = \frac{1}{(n+1)^{n+1}}$ finishes the proof under the smoothness assumption above. The general case is proved in a similar way, but using the singular variant of 2.2 proved in [1] (Theorem 3.1); compare the proof of Lemma 3.4. To prove the last statement we

recall the *subextension theorem* [22] saying that given Ω and $\tilde{\Omega}$ two hyperconvex domains such that $\Omega \subset \tilde{\Omega}$ and a function $u \in \mathcal{F}(\Omega)$ there is a function $\tilde{u} \in \mathcal{F}(\tilde{\Omega})$ such that $\tilde{u} \leq u$ on Ω and $\int_{\tilde{\Omega}} MA(\tilde{u}) \leq \int_{\Omega} MA(u)$ (up to taking approximations \tilde{u} is obtained by solving the Dirichlet problem $MA(\tilde{u}) = 1_{\Omega}MA(u)$ on $\tilde{\Omega}$). Applying subextension to $\Omega \subset r(\mathcal{B} \times D)$ for r sufficiently large thus shows that the sharp B-M inequality holds on any hyperconvex domain Ω . Finally, if we instead assume that the quasi-sharp M-T holds in dimension $n-1$ and take u such that $\mathcal{M}(u) = 1$ then repeating the same argument gives, with $v = (n+1-\delta)\mathcal{E}(u(t, \cdot))$ that

$$\int_D \left(\int_{\Omega} e^{-(n+1-\delta)u(t,z)} dV(z) \right) dV(t) \leq C' \delta^{-n} \left(1 - \frac{(n+1-\delta)^n}{(n+1)^n} \right)^{-1}$$

and expanding $1 - t^n = (1-t)(1 + \dots + t^n)$ then concludes the proof. \square

4.2. Quasi B-M in \mathbb{C}^n implies quasi M-T in \mathbb{C}^n and the free energy functional. In this section it will be convenient to use a different normalisation of \mathcal{E} obtained by multiplication by $n!$, i.e. we let

$$\mathcal{E}(u) := \frac{1}{n+1} \langle u, (dd^c u)^n \rangle, \quad \langle u, \mu \rangle := \int_{\Omega} u \mu$$

With this new normalization $d\mathcal{E}|_u = (dd^c u)^n$ and the sharp M-T inequality may be formulated as $\int e^{-(n+1)u} \leq C e^{-(n+1)\mathcal{E}(u)}$.

Proposition 4.2. *If the quasi-sharp Brezis-Merle inequality holds on $\Omega \subset \mathbb{C}^n$ then so does the quasi-sharp Moser-Trudinger inequality.*

The proof uses the “thermodynamical formalism” recently introduced in the setting of compact Kähler manifolds in [9]. The key point is to show that, by Legendre duality, the (sharp) Moser-Trudinger inequality is equivalent to yet another inequality, namely one which coincides with the classical *logarithmic Hardy-Sobolev (LHS) inequality* when $n = 1$. To make this precise we first define, for any given positive number γ ,

$$\mathcal{G}_{\gamma}(u) := \mathcal{E}(u) - \mathcal{L}_{\gamma}(u), \quad \mathcal{L}_{\gamma}(u) = -\frac{1}{\gamma} \log \int_X e^{-\gamma u} dV,$$

where $u \in \mathcal{E}^1(\Omega)$ so that \mathcal{G}_{γ} is bounded from above for $\gamma = n+1$ precisely when the sharp Moser-Trudinger inequality holds. As for the LHS type inequality referred to above it is said to hold when the following *free energy functional* F_{γ} is bounded from above:

$$F_{\gamma}(\mu) := E(\mu) - \frac{1}{\gamma} D(\mu)$$

where μ is a probability measure on $\bar{\Omega}$ with $E(\mu) < \infty$, where $E(\mu)$ is the (pluricomplex) energy of μ and $D(\mu)$ is its relative entropy, whose definitions we next recall. Following [20] a measure μ on Ω is said to have finite (*pluricomplex energy*) $E(\mu)$ if it admits a finite energy potential u_{μ} , i.e. $u_{\mu} \in \mathcal{E}^1(\Omega)$ and

$$(4.2) \quad (dd^c u_{\mu})^n = \mu$$

One may then define its energy by

$$E(\mu) := -\frac{n}{n+1} \langle u_{\mu}, \mu \rangle$$

which is finite and non-negative (the reason for our normalization appears in formula 4.4 below). If u_{μ} does not exist one sets $E(\mu) = \infty$. We also recall the

classical notion of relative entropy: given a measure μ its *relative entropy* (wrt dV) is defined as

$$D(\mu) := \int \log(\mu/dV) \mu$$

if μ is a probability measure which is absolutely continuous wrt dV (with density μ/dV) and otherwise $D(\mu) := \infty$. To see the relation to the Moser-Trudinger inequality we recall that E and $\frac{1}{\gamma}D$ can be realized as Legendre type transforms of the concave functional \mathcal{E} and \mathcal{L}_γ , respectively. Indeed, it is a classical fact (see [9] and references therein) that

$$(4.3) \quad \frac{1}{\gamma}D(\mu) = \mathcal{L}_\gamma^*(\mu) := \sup_{u \in C^0(X)} \left(-\frac{1}{\gamma} \log \int_X e^{-\gamma u} \mu_0 - \langle u, \mu \rangle \right)$$

Moreover, it follows from the concavity of \mathcal{E} and the solvability of equation 4.2 that ¹

$$(4.4) \quad E(\mu) = \sup_{u \in \mathcal{E}^1(\Omega)} \mathcal{E}(u) - \langle u, \mu \rangle$$

The idea is now to first show that

$$(4.5) \quad F_\gamma \leq C_\gamma \implies \mathcal{G}_\gamma := \mathcal{E} - \mathcal{L}_\gamma \leq C_\gamma$$

and then prove that $F_\gamma \leq C_\gamma$ for $\gamma < n + 1$, giving the desired M-T inequality. If E were a proper Legendre transform of \mathcal{E} (i.e. if the sup in 4.4 could be taken over $C^0(\bar{\Omega})$) then 4.5 would follow immediately from the fact that the Legendre transform is involutive together with the trivial implication

$$f \leq g + C \implies f^* \leq g^* + C (\implies f \leq g + C)$$

In the Kähler setting it was explained how to use a certain projection operator P to realize E the Legendre transform of $\mathcal{E} \circ P$, but for the implication 4.5 this will not be needed. Indeed, by the concavity of \mathcal{E} on $\mathcal{E}^1(\Omega)$ we have, for any fixed measure μ ,

$$(4.6) \quad \mathcal{E}(u) \leq \mathcal{E}(u_\mu) + \langle u - u_\mu, \mu \rangle = E(\mu) + \langle u, \mu \rangle = F_\gamma(\mu) + \left(\frac{1}{\gamma}D(\mu) + \langle u, \mu \rangle \right)$$

The proof may now be concluded by noting that (compare 4.3)

$$\inf_{\mu} \left(\frac{1}{\gamma}D(\mu) + \langle u, \mu \rangle \right) = \mathcal{L}_\gamma(u)$$

where the infimum is taken over all measures on Ω . More concretely, we may by approximation, assume that $u \in \mathcal{H}_0(\Omega)$ and then note that $\mu = e^{-\gamma u} / \int e^{-\gamma u} dV$ realizes the inf above in 4.6, so that the previous argument gives

$$\mathcal{G}_\gamma(u) \leq F_\mu(e^{-\gamma u} / \int e^{-\gamma u} dV) \leq C_\gamma$$

proving 4.5.

Finally, to estimate F_γ we next define the following general invariant of a pair (Ω, μ_0) where μ_0 is a measure on Ω :

$$(4.7) \quad \alpha(\Omega, \mu_0) := \sup \left\{ t : \exists C_t; \int_\Omega e^{-tu} d\mu_0 \leq C_t \ \forall u \in \mathcal{H}_0(\Omega)_b \cap \left\{ \int_\Omega (dd^c u)^n / n! = 1 \right\} \right\}$$

¹In fact, using a variational approach the potential u_μ above may be obtained directly by maximizing the functional in the rhs of 4.4. This was recently shown in the Kähler setting in [10] and in the setting of domains in [2] (compare section 7).

Lemma 4.3. *If $\gamma < \alpha \frac{(n+1)}{n}$, then $F_\gamma(\mu)$ is bounded from above, i.e. $F_\gamma(\mu) \leq C_\gamma$. More precisely, for any $t < \alpha$*

$$F_\gamma(\mu) \leq \left(\frac{t}{\gamma} - \frac{n}{n+1}\right) \langle u_\mu, \mu \rangle + \frac{t}{\gamma} C_t$$

where C_t is the minimum of $\mathcal{L}_t(u)$ over all $u \in \mathcal{H}_0(\Omega)_b \cap \{\int_\Omega (dd^c u)^n = 1\}$.

Proof. Given γ we fix $t < \alpha := \alpha(\Omega, \mu_0)$. By the definition of α we have $\mathcal{L}_t(u) \geq -C_t$ if $u \in \mathcal{H}_0(\Omega) \cap \{\int_\Omega (dd^c u)^n = 1\}$ and hence

$$\frac{1}{t} D(\mu) = \mathcal{L}_t^*(\mu) \geq \mathcal{L}_t(u_\mu) - \langle u_\mu, \mu \rangle \geq -\langle u, \mu_\mu \rangle - C_t$$

As a consequence

$$F_\gamma(\mu) \leq \left(-\frac{n}{n+1} + \frac{t}{\gamma}\right) \langle u_\mu, \mu \rangle + t C_t$$

Given γ such that $\gamma < \alpha \frac{(n+1)}{n}$, we may now choose t sufficiently close to α so that the multiplicative constant above is strictly positive, thus concluding the proof. \square

Assume now that the quasi-sharp BM-inequality holds in Ω . The point is that this implies that $\alpha(\Omega, dV) = n$ and the previous Lemma then shows that $F_{n+1-\delta}$ is bounded from above. However, we can actually be more precise wrt the depends on δ . Indeed, according to the formulation 4.1 we have that $C_{n-\epsilon} \leq C + \log(1/\epsilon^{n-1})$ where C_t is defined as in the previous lemma (with $\mu_0 = dV$). Applying the previous lemma with $\gamma = n+1-\delta$ and $t = n - \delta/2$ hence gives

$$F_{n+1-\delta}(\mu) \leq C_{n-\delta/2} \leq C' + \log(1/\delta^{n-1})$$

The proof of Prop 4.2 is now concluded by using 4.5.

Remark 4.4. When $\mu_0 = dV$ is any volume form on $\bar{\Omega}$ $\alpha := \alpha(\Omega, dV)$ defines an invariant of a domain Ω which can be seen as a variant of Tian's α -invariant for a Kähler manifold (X, ω) (or rather the class $[\omega]$). The difference is that in the latter case the Monge-Ampère mass is, of course, determined by $[\omega]$ and hence independent of u . In this latter setting $-\gamma F_\mu(MA(u))$ coincides with Mabuchi's K-energy functional, which plays a key role in Kähler geometry (compare the discussion in [9])

4.3. Proof of Theorem 1.5. The sharp Moser-Trudinger inequality holds when $n = 1$ in the disc D [45]. Hence combining Prop 4.1 and Prop 4.1 simultaneously prove the inequalities in Theorem 1.5 and Theorem 1.6.

As for the sharpness of the multiplicative constants in inequalities we make the following remark which concludes the proof of Theorem 1.5...

Remark 4.5. Let $\Omega := \mathcal{B}$ be the unit-ball in \mathbb{C}^n and set $u := \log |z|^2$ so that $(dd^c u)^n = \delta_0$. Letting $u_t := tu$ for $t < 1$ gives $\int_{\mathcal{B}} e^{-u_t} = \frac{1}{1-t/n} \sim \frac{1}{(1-\frac{t}{n})^n}$ as $t \rightarrow 1^-$. Moreover, since $MA(u_t) = t^n$ this shows that the sharp Brezis-Merle inequality cannot hold on \mathcal{B} with a better coefficient than $\frac{1}{n^n}$, nor with a smaller power in the rhs. Using the subextension theorem (see the proof of Theorem) gives the same conclusion for any hyperconvex domain Ω (alternatively when can apply the same argument with u replaced by the pluricomplex Green function g_z with a pole at any fixed point z in Ω). Finally, by Prop 4.1 this also shows that the coefficient $n!/(n+1)^n$ in the sharp M-T inequality cannot be improved for any hyperconvex domain Ω .

5. RELATIONS BETWEEN THE VARIOUS INEQUALITIES

Let $u \leq 0$ be a, say continuous, function on a topological space X and dV a finite measure on X . We let

$$E(t) := \int e^{-tu} dV$$

and

$$V(s) := \text{Vol} \{u < -s\} := \int_{\{u < -s\}} dV$$

Then $E(t)/t$ and $V(s)$ are (up to signs) related by Laplace transforms. Indeed, by the push-forward formula and integration by parts

$$E(t) := t \int_0^\infty e^{ts} V(s) ds$$

According to a well-known principle the Laplace transform is asymptotically described by the Legendre transform:

$$E(t) \lesssim e^{f(t)} \quad \Longleftrightarrow \quad V(s) \lesssim e^{-f^*(s)}$$

(as t and s tend to infinity), where f is assumed convex and $f^*(s)$ is its Legendre transform:

$$f^*(s) := \sup_t (st - f(t))$$

There are various ways of formulating this principle precisely but for our purposes the following basic lemma will be sufficient:

Lemma 5.1. *If $E(t) \leq Ce^{f(t)}$, then $V(t) \leq Ce^{-f^*(s)}$. Conversely, if $V(t) \leq Ce^{-g(s)}$ then for any $\delta > 0$ there is a constant C_δ such that $E(t) \leq C_\delta e^{g^*(t+\delta)}$.*

Proof. Fix $t \in \mathbb{R}$. On the subset $\{u < -s\}$ of X we have $1 < e^{-st}e^{-tu}$ and hence $V(s) \leq e^{-st} \int_X e^{-tu} dV \leq Ce^{-st+f(t)}$. Taking the infimum over all t then proves the first inequality. The second inequality follows immediately from the definitions if we rewrite $ts - g(s) = ((t+\delta)s - g(s)) - \delta s$ and let $C_\delta = C \int_0^\infty e^{-\delta s} ds = C/\delta$. \square

We will apply the previous lemma to the case when $f(t)$ is homogeneous and use the following basic relations (assuming $p > 1$)

$$(5.1) \quad f(t) = \frac{1}{a} s^p / p \quad \Longleftrightarrow \quad f^*(s) := a^{(q-1)} t^q / q$$

where $1/p + 1/q = 1$ (the case $a = 1$ is immediate and implies the general case by scaling). More precisely, in our case we will have $p = (n+1)/n$ and hence $q = n+1$ and vice versa.

Corollary 5.2. *(of Theorem 1.1): Let (X, ω) be a compact Kähler manifold and $u \in \mathcal{H}_0(X, \omega)$. Then there are constants A and B such that*

$$\text{Vol}_\omega \{u < -s\} \leq Ce^{-B \frac{1}{(-\mathcal{E}_\omega(u))^{1/n}} s^{(n+1)/n}}$$

More precisely, we may replace the exponent above by

$$-\frac{n}{(-\mathcal{E}_\omega(u))^{1/n}(n+1)^{(1+1/n)}} s^{(n+1)/n} (1 + o(1))$$

as $s \rightarrow \infty$.

From the first volume estimate in the previous corollary we see that the L^p -norms of u may be estimated as

$$\int_X (-u)^p dV = \int_0^\infty V(s) d(s^p) \leq C \Gamma\left(\frac{n}{n+1} p\right) \left(\frac{1}{B}\right)^{pn/(n+1)} (-\mathcal{E}_\omega(u))^{p/(n+1)}$$

(after setting $x = s^{(n+1)/n}$ and using $\Gamma(x)x = \Gamma(x+1)$, for $\Gamma(x) := \int_0^\infty s^{x-1}e^{-s}ds$). Using $\Gamma(m) = (m-1)!$ and Stirling's approximation $m! \sim (m/e)^m$ hence gives the Sobolev type inequality 1.6 from the introduction.

The inequality 1.5 can now be deduced from the previous Sobolev type inequality (compare [52]). Indeed, assuming first that $-\mathcal{E}_\omega(u) = 1$ gives

$$\int e^{B(1-\delta)(-u)^{n+1/n}} dV = \sum_{p=1}^{\infty} \frac{B^j}{j!} \int_X (-u)^{j(n+1)/n} dV \leq \sum_{p \in \mathbb{N}(n+1)/n} \frac{1}{p} (1-\delta)^{pn/(n+1)}$$

which is finite for any $\delta > 0$ and the general case then follows by scaling. Note in particular, that when $E(t) \leq e^{At^{n+1}}$ with $A = (n+1)^{-(n+1)}$ then $V(s) \leq e^{-Bs^{(n+1)/n}}$ with $B = n$ which proves the last statement in Theorem 1.5.

6. REMARKS ON THE OPTIMAL CONSTANTS

In this section we will compare our results with Aubin's conjectures [3, 4] (and partial results). To this end we first have to compare our notations, which differ slightly. There are two reasons for the differences which come from (1) the choice of energy functional (2) the normalizations of the energy functional. We start with the energy functionals (in our normalizations). Given the functional \mathcal{E}_ω which we recall may be defined as a primitive of the Monge-Ampère operator one defines

$$J_\omega(u) := -\mathcal{E}_\omega(u) + \int u\omega^n/n!$$

and

$$I_\omega(u) := \frac{1}{n!} \int (-u) (\omega_u^n - \omega^n)$$

In particular, the functionals J_ω and I_ω are both \mathbb{R} -invariant and semi-positive [3] and when $\omega = 0$ (as in the \mathbb{C}^n -setting) they coincide. In general, they are equivalent up to multiplicative factors [3]:

$$J_\omega \leq I_\omega \leq (n+1)J_\omega$$

However, Aubin's normalizations are slightly different and obtained by replacing the factor $1/n!$ above by $(2\pi)^n/(n-1)!$. In particular,

$$(-\mathcal{E}_\omega) = d_n J_\omega^{(A)}, \quad d_n := \frac{1}{n} \frac{1}{(2\pi)^n}$$

if $\int u\omega = 0$, where the super script A refers to Aubin's normalizations. In this notation Aubin's general "Hypothèse fondamentale" as formulated in [3] asserts that there exist positive constants ξ and C such that

$$(6.1) \quad \int e^{-ku} dV \leq C \exp(\xi k^{n+1} I_\omega^{(A)}(u))$$

for all $u \in \mathcal{H}(X, \omega)$ normalized such that $\int_X u\omega^n = 0$. To see that Theorem 1.1 confirms this conjecture (in the case when $[\omega]$ is an integral class) we recall that there is a constant C' such that

$$(6.2) \quad \sup u \leq \frac{1}{V} \int u\omega^n + C'$$

and hence 6.1 applied to $u - \sup u$ gives

$$\log \int e^{-ku} dV \leq Ak^{n+1} \left(J_\omega(u) + \int (-u) \frac{\omega^n}{n!} \right) + (AC'k^{n+1} + B)$$

Thus 6.1 holds with $\xi = Ak^{n+1}$ and $C = C_k := \exp(AC'k^{n+1})$. This means that the constant C_k depends on k while Aubin's hypothesis, strictly speaking, says that it should be independent of k . Anyway, in applications to existence problems for PDEs the precise value of C_k is immaterial (compare section 7).

6.1. Counter-example to Aubin's explicit conjecture in the Fano case.

In his paper Aubin also conjectured that in the Fano setting (with $[\omega] = c_1(-K_X)$) the infimum ξ_n over all constants ξ satisfying 6.1 for some C_ξ is explicitly given by

$$(6.3) \quad \xi_n = \pi^{-n}(n-1)!n^n(n+1)^{-(2n+1)} = \pi^{-n}(n-1)! \left(1 + \frac{1}{n}\right)^{-n} (n+1)^{-(n+1)}$$

However, there is a simple counter-example to this hypothesis. To see this we first recall a result of Ding ([29], Prop 6) which in our notation may be formulated as follows: if one replaces I_ω in 6.1 by $(-\mathcal{E}_\omega)$ then the corresponding optimal constant $\eta(X)$ satisfies (when specializing to the case $k = 1$)

$$\eta(X) \geq 1/V(X)$$

if X admits non-trivial holomorphic vector fields, i.e. if $H^0(TX) \neq \{0\}$. To get a contradiction it will hence be enough to exhibit a Fano manifold X_n with $H^0(TX_n) \neq \{0\}$ such that

$$(6.4) \quad V(X_n) < \frac{1}{(n+1)} \frac{d_n}{\xi_n} \left(= \frac{1}{(n+1)!} \frac{1}{2^n} \left(1 + \frac{1}{n}\right)^n (n+1)^{(n+1)} \right)$$

To this end we may simply set $X_n = (\mathbb{P}^1)^n$ so that $c_1(X_n)^n = 2^n$. Since $V(X_n) := c_1(X_n)^n/n!$ the previous inequality indeed holds for all sufficiently large n . Indeed, when dividing out $n!$ the lhs in 6.4 is equal to 2^n while the rhs is of the order $(n/2)^n$.

6.2. Comparison with Aubin's constant for the ball. Let us now turn to the setting of the unit-ball, where $\omega = 0$ and consider the corresponding functional $I_0^{(A)}$ (called \mathcal{J} with the same normalizations in [4]), i.e.

$$I_0^{(A)}(u) := \frac{1}{(n-1)!} \int (-u) ((i\partial\bar{\partial}u)^n$$

In the case of a radial psh function u in the ball Aubin showed [4] that

$$(6.5) \quad \log \int e^{-u} dV \leq a_n I_0^{(A)}(u) + C, \quad a_n = 2n^n(n+1)^{-(2n+1)} \sigma_{2n-1}^{-1},$$

where σ_p denotes the volume of the unit p -sphere, giving $a_n = \xi_n$ (formula 6.3). But by Theorem 1.5 the *optimal* constant c_n in the equality 6.5 is equal to

$$c_n = \frac{1}{(2\pi)^n} \frac{(n-1)!}{(n+1)!} \frac{n!}{(n+1)^n} = \frac{(n-1)!}{(2\pi)^n} \frac{1}{(n+1)^{n+1}}$$

Hence,

$$c_n = \left(\frac{1}{2} (1 + 1/n) \right)^n a_n,$$

so that $a_n \geq c_n$ with equality iff $n = 1$. Accordingly, Aubin's constant a_n is not optimal for $n > 1$.

6.3. Discussion. It is natural to ask why Aubin expected that the particular value in formula 6.3 gives the optimal constant in the Fano case? We can only speculate on this. But it seems that Aubin was expecting that the optimal constant in the Fano case coincides with the optimal constant in the setting of the ball. In fact, in section 3 in [3] Aubin claims that he has proved that the optimal constant in the setting of the ball is indeed given by formula 6.3. But as explained in the previous section this is *not* the optimal constant in the ball unless $n = 1$ (and moreover Aubin only proved his inequality in the radial case). In particular, it is not the case that the optimal constant in the Fano case coincides with the case of the ball (by the counter-example above).

It may be illuminating to give an informal description of counter-examples to Aubin's expectations (which has the virtue of avoiding comparing various normalizations). As our arguments used in the proof of Theorem 1.4 show, the optimal constant in the setting of the ball coincides with the optimal constant in the Fano setting for $X = \mathbb{P}^n$. But by Ding's result the optimal constant $\xi(X)$ on any Fano manifold X with $H^0(TX) \neq \{0\}$ satisfies

$$\xi(X) \geq C_n/V(X)$$

where C_n is a universal constant (depending on the particular normalizations of the energy functionals) with equality if X moreover admits a Kähler-Einstein metric (by [30]). Hence, if the optimal constant coincided with the one in the setting of the ball, then this would force

$$V(X) \geq V(\mathbb{P}^n)$$

for any Fano X such that $H^0(TX) \neq \{0\}$. But this latter inequality is clearly violated by $X = (\mathbb{P}^1)^n$ and in fact by many other X . For example, according to two well-known conjectures \mathbb{P}^n is the unique maximizer of the volume functional among (1) all Fano n -folds with Picard number equal to one (see [36] for a proof when $n \leq 4$) and (2) among all toric Kähler-Einstein Fano manifolds (such as $(\mathbb{P}^1)^n$).

7. EXISTENCE OF EXTREMALS AND APPLICATIONS TO MONGE-AMPÈRE EQUATIONS

7.1. The Kähler setting. Let (X, ω) be an integral Kähler manifold and fix a smooth volume form dV on X . For a given sequence $a_k \in \mathbb{R}$ we consider the following Moser-Trudinger type functional on $\mathcal{H}(X, \omega)$:

$$\mathcal{G}_{a_k}(u) := \frac{1}{k} \log \int e^{-ku} dV + \frac{1}{V} \int u \frac{\omega^n}{n!} - \frac{k^n}{a_k} J_\omega(u)$$

which is \mathbb{R} -invariant (and hence descends to a functional on space of all Kähler metrics in $[\omega]$). We let $a_k(X)$ be the infimum over all a_k such that the functional above is bounded from above. By Theorem 1.1 (and the discussion in the beginning of section 6) $a_k(X) \geq 1/A$ or more precisely $\liminf_k a_k(X)/k^{n+1} \geq 1$.

In this section we will be concerned with the question of existence of maximizers for \mathcal{G}_{a_k} and solutions to the corresponding Euler-Lagrange equation

$$(7.1) \quad 0 = (d\mathcal{G}_{a_k})|_u = -\frac{e^{-ku} dV}{\int e^{-ku} dV} + \frac{1}{V} \frac{\omega^n}{n!} + \frac{k^n}{a_k} \left(\frac{\omega_u}{n!} - \frac{\omega^n}{n!} \right)$$

Braking the \mathbb{R} -invariance by the introducing the normalization $\int_X e^{-ku} dV = V$ the previous equation can hence be written as the following PDE:

$$(7.2) \quad \frac{\omega_u^n}{n!} = \frac{a_k}{k^n} e^{-ku} dV + \left(1 - \frac{a_k}{Vk^n}\right) \frac{\omega^n}{n!}$$

for $u \in \mathcal{H}(X, \omega)$.

Theorem 7.1. *If $a_k < a_k(X)$ and $a_k < Vk^n$ then there is a solution to 7.2 in $\mathcal{H}(X, \omega)$. Moreover, the solution can be taken to maximize the functional \mathcal{G}_{a_k} . In particular, if $a_k = a < 1$ then there is such a solution for all k sufficiently large.*

Given the Moser-Trudinger inequalities in Theorem 1.1 the proof of the previous theorem follows from the variational approach to complex Monge-Ampère equation introduced in [10].

Existence of a maximizer u_ in $\mathcal{E}^1(X, \omega)$*

We proceed in two steps. The first step amounts to the following *coercivity estimate*: there exists $\delta, C > 0$ such that

$$(7.3) \quad \mathcal{G}_{a_k} \leq \delta \mathcal{E}_\omega + C$$

on the space $\mathcal{E}_0^1(X, \omega) := \mathcal{E}^1(X, \omega) \cap \{\sup_X = 0\}$ (which we equip with the L^1 -topology). This follows directly from the assumption that $a < a(X)$ and the inequality 6.2. The second step is to establish the following *semi-continuity property*: for any constant C the functional \mathcal{G}_{a_k} is upper semi-continuous (usc) on $\{-\mathcal{E}_\omega \leq C\}$ in $\mathcal{E}_0^1(X, \omega)$ (wrt the L^1 -topology). To this end first recall that \mathcal{E}_ω is usc on $PSH(X, \omega)$ (in particular it follows from weak compactness that $\{-\mathcal{E}_\omega \leq C\}$ is compact) [10, 16]. All that remains is then to prove that $u \mapsto \int e^{-ku} dV$ is usc on $\{-\mathcal{E}_\omega \leq C\}$. To this end it is enough to establish a uniform bound

$$(7.4) \quad \int e^{-(k+\delta)u} \leq C_\delta$$

for some $\delta > 0$ (compare the proof of Lemma 6.4 in [10] or Lemma 3.6 in [9]). But since we have assumed that $u \in \{-\mathcal{E}_\omega \leq C\}$ this is an immediate consequence of the Moser-Trudinger inequality in Theorem 1.1 (which shows that any $\delta > 0$ will do). The existence of a maximizer u_* is now rather immediate: take u_j in $\mathcal{E}_0^1(X, \omega)$ such that

$$\mathcal{G}_{a_k}(u_j) \rightarrow \sup_{\mathcal{E}^1(X, \omega)} \mathcal{G}_{a_k},$$

(note that, by the scale invariance of \mathcal{G}_{a_k} we may indeed assume that $\sup_X u_j = 0$). By the coercivity estimate the sup is finite and moreover $(u_j) \subset \{-\mathcal{E}_\omega \leq C\}$ for some $C > 0$. But then it follows from the upper semi-continuity that the sup is attained on any accumulation point u_* of (u_j) (which exists by compactness). This concludes the proof of the existence of a maximizer.

The maximizer u_ is a weak solution of equation 7.2.*

We will use the projection argument in [16] to see that u_* is a (weak) solution in $\mathcal{E}_0^1(X, \omega)$ to the variational equation 7.1 (shifting u_* by a constant hence gives a solution to the equation 7.2). To this end we first decompose

$$\mathcal{G}_{a_k}(u) = \frac{k^{n+1}}{a_k} \mathcal{E}_\omega + \mathcal{I}_{a_k}, \quad (\mathcal{I}_{a_k}(u) = \log \int e^{-ku} dV + k(1 - \frac{k^n}{V a_k}) \int u \omega^n / n!)$$

Fixing $v \in C^\infty(X)$ let $f(t) := \mathcal{E}_\omega(P_\omega(u_* + tv) + \mathcal{I}_{a_k}(u_* + tv)$, where

$$P_\omega(u)(x) := \sup\{v(x) : v \leq u, v \in PSH(X, \omega)\}$$

By the assumption $a_k < k^n V$ the functional $\mathcal{I}_{a_k}(u)$ is decreasing in u and hence the sup of $f(t)$ on \mathbb{R} is attained for $t = 0$. Now $\mathcal{E}_\omega \circ P_\omega$ is differentiable with differential $MA(P_\omega u)$ at u [10]. Hence, the condition $df/dt = 0$ for $t = 0$ gives that the variational equation 7.1 holds when integrated against any $v \in C^\infty(X)$.

Regularity

Now, by the previous estimate 7.4 $\omega_{u_*}^n$ has a density in L^p for some $p > 1$ (or even all $p > 1$) and hence it follows from Kolodziejs L^∞ -estimate [42] that u_* is in $L^\infty(X)$ (and is even continuous). Finally the higher order regularity $u \in C^\infty(X)$ then follows from [47], using that the rhs in equation 7.2 is of the form $F(u)$ for $F(t)$ smooth and positive (using the assumption $a_k < k^n V$).

7.2. Remarks on the Fano setting. Let now X be Fano with $[\omega] = c_1(-K_X)$. In the case when $k = 1$ and $a_k := V$ the functional \mathcal{G}_{a_k} above becomes

$$\mathcal{G}_{a_k} := \mathcal{G}_V(u) := \log \int e^{-u} dV + \frac{1}{V} \mathcal{E}_\omega(u)$$

with Euler-Lagrange equation

$$\omega_u^n/n! = V e^{-u} dV$$

In particular, if dV is taken as $e^{-h_\omega} \omega^n/n!$ where h_ω is the Ricci potential of ω then the previous equation may be written as the Kähler-Einstein equation

$$(dd^c \phi)^n/n! = V e^{-\phi} dz \wedge d\bar{z}$$

for the local weight ϕ of the metric ω , saying that $\text{Ric } \omega = \omega$. In this setting it is well-known that the corresponding coercivity estimate 7.3 is *equivalent* to the existence of a Kähler-Einstein metric, which in turn is equivalent to X being “analytically K -stable” in the sense of Tian (which means that Mabuchi’s K-energy functional is proper); see [50] Thm 7.13 and [46].

Now, the coercivity estimate holds for \mathcal{G}_V precisely when a Moser-Trudinger inequality holds for some $a_k := a$ (i.e. $\mathcal{G}_a \leq C$) satisfying

$$(7.5) \quad V < a$$

In other words, if a could be chosen uniformly over all Fano manifolds X of dimension n then the previous inequality would give an existence criterion for Kähler-Einstein metrics on X , in terms of the volume of X . This follows for example from the variational approach above, but a proof using the continuity method already appears in Aubin’s paper [3] (see also [29] where the functional \mathcal{G}_V seems to first have appeared explicitly). As explained in section 6 Aubin also proposed an explicit value for a , which however cannot be correct.

Unfortunately, it can be shown that the uniform constant provided by Theorem 1.2 (at least in its present form) is not useful for this kind of application. On the other hand the existence of Moser-Trudinger type inequalities established in Theorems 1.1 and 1.2 are very useful in other regards, for example for establishing semi-continuity properties and uniform estimates as in the previous section. In particular, it plays an important role in [11] in the construction of Kähler-Einstein metrics on “analytically K -stable” log-Fano varieties.

Before turning to the setting of domains in \mathbb{C}^n we briefly recall Tian’s [48] existence criterion for Kähler-Einstein metrics which has proved to be very useful:

$$(7.6) \quad \alpha(X) > n/(n+1),$$

where

$$\alpha(X) := \sup \left\{ t : \exists C_t : \int_X e^{-t(u - \sup_X)} dV \leq C_t, \forall u \in PSH(X, \omega) \right\}$$

As is well-known it is enough to consider u with analytic singularities in the sup above (and hence $\alpha(X)$ coincides with the algebraically defined log canonical threshold $\text{lc}(X)$). Now, if it would be enough to take the sup above over all u with *isolated* singularities, then it would follow from the inequality 1.13 (see also below) that

$$\alpha(X) > n/(n!V)^{1/n}$$

and hence Tian’s criterion 7.6 would be satisfied if $n!V < (n+1)^n$. However, this latter condition is satisfied for *any* Fano manifold when $n = 2$ (i.e. Del Pezzo surfaces) and in particular for those which do not admit a Kähler-Einstein metric (like \mathbb{P}^2 blown-up in one point). Still, as we will see next a similar approach turns out to be very fruitful in the setting of domains. At least on a heuristic level this could perhaps be expected as all analytic singularities are indeed isolated in this setting.

7.3. The setting of domains in \mathbb{C}^n and Mean Field Equations. Let now Ω be a hyperconvex domain in \mathbb{C}^n with dV the Euclidean volume form and recall (see section 4.2) that

$$\mathcal{G}_\gamma(u) := \frac{1}{\gamma} \log \int_{\Omega} e^{-\gamma u} dV + \frac{1}{n+1} \int (-u)(dd^c u)^n$$

so that the corresponding Euler-Lagrange equation reads

$$(7.7) \quad (dd^c u)^n = \frac{e^{-\gamma u} dV}{\int_{\Omega} e^{-\gamma u} dV}$$

with the boundary condition $u = 0$. Equivalently setting $v = \gamma u$ gives the Euler-Lagrange equation corresponding to the non-scaled Moser-Trudinger inequality $(M - T)$ in the beginning of section 4 (it is obtained by setting $\gamma = 1$ and inserting a multiplicative constant $a = \gamma^n$ in the rhs). Ideally, we would like to look for smooth solutions (in $\mathcal{H}_0(\Omega)$) to the previous equation, but as the corresponding higher order regularity theory does not seem to be sufficiently developed we will merely be able to produce continuous solutions (vanishing on the boundary). Note that in this setting there is no invariance under additive scalings of u (due to the boundary conditions $u = 0$).

In the case when $n = 1$ the previous equation is often referred to as the *mean field equation* as it appears in a statistical model of mean field type, with γ playing the role of (minus) the temperature [18, 38]. In the one-dimensional case it is well-known that $\gamma = 2$ appears as a critical value/phase transition (the value is 8π when dd^c is replaced by the usual non-normalized Laplacian in the plane). It should be emphasized that the statistical mechanical point of view only the solutions maximizing the corresponding free energy functional are relevant.

Theorem 7.2. *Let Ω be a hyperconvex domain and assume that $\gamma < n + 1$. Then there exists $u_\gamma \in \mathcal{C}^0(\bar{\Omega})$ solving equation 7.7 in Ω with $u_\gamma = 0$ on $\partial\Omega$ and which maximizes the corresponding functional \mathcal{G}_γ .*

Proof. Assume that $\gamma < n + 1$. By Theorem 1.5 the coercivity estimate corresponding to 7.3 still holds for \mathcal{G}_γ and it is well-known that \mathcal{E} is usc and its sub-level sets $\{\mathcal{E} \geq -C\}$ are compact (wrt the L^1_{loc} -topology); see [2] and references therein. Hence, all the previous arguments still apply in the present setting of domains to give the existence of a maximizer u_γ for \mathcal{G}_γ on the space $\mathcal{E}^1(\Omega)$. To see that u_γ satisfies the equation 7.7 one applies a projection argument as in the Kähler setting above (see [2] where the projection argument from [10] was adapted to the setting of hyperconvex domains). Finally, by the M-T inequality $MA(u)$ has an L^p -density for $p > 1$ and hence when Ω is strictly pseudoconvex the continuity statement follows from [42], or alternatively from [23] by taking $p = 2$ (using the uniqueness of solution to the inhomogeneous Monge-Ampère equation in the class $\mathcal{E}_1(\Omega)$). As for the general hyperconvex case it follows from [15]. \square

Next we will establish a “concentration/compactness principle” for the behavior of the solutions above when γ approaches the critical value $n + 1$. First recall that if u psh in a neighborhood of a point z_0 then its *complex singularity exponent* $c_{z_0}(u)$ at z_0 is defined as

$$c_{z_0}(u) := \sup \left\{ t : \int_U e^{-tu} dV < \infty \right\},$$

for U some neighborhood of z_0 . As shown in [1] (Thm 5.5) the Brezis-Merle type inequality proved there may be localized to give that

$$c_{z_0}(u) \geq n / \left(\int_{\{z_0\}} (dd^c u)^n \right)^{1/n}$$

for any point $z_0 \in \Omega$ and function $u \in \mathcal{F}(\Omega)$ (more generally the boundary assumptions on u are not needed). This can be seen as a generalization of the local algebra inequality 1.13 which corresponds to the case when $u = \log(\sum_{i=1}^m |f_i|^2)$ for holomorphic functions f_i (determining the ideal $\mathcal{I} := (f_1, \dots, f_m) \subset \mathcal{O}_{z_0}(\mathbb{C}^n)$).

Theorem 7.3. *Let γ_j be a sequence increasing to $n+1$ and $u_j := u_{\gamma_j}$ a sequence of solutions of equation 7.7 as in the previous theorem converging to $u \in \mathcal{F}(\Omega)$ in the L^1_{loc} -topology (which is always possible to find after passing to a subsequence), then precisely one of the following two alternatives hold:*

- (1) u_j converges uniformly to a solution u of equation 7.7 for $\gamma = n+1$, maximizing the functional \mathcal{G}_{n+1} .
- (2) For any $\delta > 0$ the sequence $\int_{\Omega} e^{-(n+\delta)u_j} dV$ is unbounded.

In the second case above either the sequence u_j has a blow-up point in $\partial\Omega$, i.e. there is a sequence of point z_j in Ω converging to $z_0 \in \partial\Omega$ such that $\lim_j u(z_j) = -\infty$ or the limit u satisfies

$$(7.8) \quad (dd^c u)^n = \delta_{z_0}$$

for some point $z_0 \in \Omega$ and moreover $c_{z_0}(u) = n$

Proof. We will use the notation from section 4.2. First note that since $\gamma \mapsto \mathcal{L}_{\gamma}$ is decreasing we have that $\mathcal{G}_{\gamma} \leq \mathcal{G}_{\gamma_*}$ if $\gamma < \gamma_*$ and in particular $\sup \mathcal{G}_{\gamma} \leq \sup \mathcal{G}_{\gamma_*}$. Hence, for $\gamma_1 < \gamma_i < n+1$ we get

$$(7.9) \quad -C := \mathcal{G}_{\gamma_1}(u_1) \leq \mathcal{G}_{\gamma_i}(u_i) \leq \mathcal{G}_{n+1}(u_i)$$

Now if the second alternative in the theorem does not hold for u_j then Lemma 4.3 shows that there exists δ such that the free energy functional $F_{n+1+\delta}$ is uniformly bounded from above along (u_j) and hence so are the functionals $\mathcal{G}_{n+1+\delta}$ (as explained in connection to Lemma 4.3). Combined with the lower bound 7.9 this means that

$$\mathcal{E}(u_j) \geq -C.$$

Hence, the Moser-Trudinger inequality applied to a fixed $\gamma_1 < n+1$ (i.e. the bound $\mathcal{G}_{\gamma_1} \leq C$) shows that $\int e^{-pu_j} \leq C_p$ for any $p > 0$. But then it follows from general principles (for the same reasons as in the Kähler case) that $\int e^{-pu_j} \rightarrow \int e^{-pu}$, i.e. $\|e^{-u}\|_{L^p(\Omega)} \rightarrow \|e^{-u}\|_{L^p(\Omega)}$ and even more precisely that

$$(7.10) \quad e^{-u_j} \rightarrow e^{-u}, \quad \text{in } L^p(\Omega).$$

In particular $\mathcal{L}_{n+1}(u_j) \rightarrow \mathcal{L}_{n+1}(u)$, as $j \rightarrow \infty$. Moreover, a similar argument shows that u is a maximizer of \mathcal{G}_u and hence the projection argument gives, as above, that u solves the equation 7.7. Moreover, the convergence 7.10 for $p=2$ gives that the $L^2(\Omega)$ -norm of the densities $MA(u_j)/dV - MA(u)/dV$ tend to zero and hence the stability result in [23] show that $u_j \rightarrow u$ in $L^\infty(\Omega)$.

Finally, if there is no blow-up point in $\partial\Omega$, then there is a constant M and a compact subset K of Ω such that $u \geq -M$ on $\Omega - K$ and hence $\int_K e^{-(n+\delta)u_j} dV$ is unbounded. Now, if $u_j \rightarrow u$ in L^1_{loc} , then it follows that $\int_{\Omega} (dd^c u)^n \leq 1$ (see for example the appendix in [27]). By the semi-continuity of complex singularity exponents [28] there is a neighborhood U of K such that $\int_U e^{-(n+\delta)u} = \infty$ for

any $\delta > 0$, i.e. $c_z(u) \leq n$, for any $z \in K$. But then it follows from 1.13 that for any $z \in K$

$$\int_{\{z\}} (dd^c \phi) \geq 1.$$

Since $\int_{\Omega} (dd^c u)^n \leq 1$ this forces the equation 7.8 to hold for some $z = z_0$. Moreover, since $\int_{\Omega} (dd^c u)^n \leq 1$ we already know, by the quasi-sharp B-M inequality that, for any $\delta > 0$ $e^{-(n-\delta)\phi}$ is in $L^1(\Omega)$ and hence $c_{z_0}(u) \geq n$. All in all this means that $c_{z_0}(u) = n$ and that ends the proof. \square

Remark 7.4. In the case when $n = 1$ it is well-known that there cannot be any blow-points on $\partial\Omega$ (see Prop 4 in [44]) and we expect this to be true in general. It also seems natural to conjecture that the limit u in the second alternative above coincides with the pluricomplex Green function g_{z_0} with a pole at z_0 . This would in fact follow if u were known a priori to have analytic singularities at z_0 , i.e. $u(z) = \lambda \log(\sum_{i=1}^m |f_i|^2) + O(1)$ close to z_0 , where $\lambda \in \mathbb{R}$ and f_i are holomorphic. Indeed, since $c_{z_0}(u) = n / \int_{\{z_0\}} (dd^c u)^n$ it would then follow from the equality case in the inequality 1.13 (see [25]) that $u(z) = \log |z - z_0|^2 + O(1)$ close to z_0 and hence $u = g_{z_0}$ by the comparison principle (at least if a priori $u(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$).

When $n = 1$ it is well-known that the question whether there exists solutions of equation 7.7 in the critical case $\gamma = 2$ depends on the geometry of Ω (see [18]). For example, for the disc there is no solution, while there is one for an annulus. In the case of a general n the sharpness part of Theorem 1.5 gives that, in the super critical case $\gamma > n + 1$, the functional \mathcal{G}_a is not bounded from above and in particular it has no maximizers (i.e. the last part of Theorem 7.2 cannot hold in this range). As for the critical case $\gamma = n + 1$ one would expect that there is no solution of the equation 7.7 when Ω is the ball. For radial solutions this is straight-forward to check. Indeed, an explicit calculation then reveals that, for any $\gamma < n + 1$, a radial solution u_γ is uniquely determined and hence given by $u_\gamma = \phi_0^\epsilon$ (formula 3.3) for some ϵ , where $\gamma \rightarrow n + 1$ corresponds to $\epsilon \rightarrow 0$. More over, when $\gamma = n + 1$ there is no radial solution and $u_\gamma \rightarrow (n + 1) \log |z|^2$ as $\gamma \rightarrow n + 1$ where u has infinite energy, i.e. it is not an element in $\mathcal{E}^1(\Omega)$. In fact, in the case $n = 1$ any solution is radial, as follows from the method of moving planes [33] (which also applies to the corresponding equation associated to the *real* Monge-Ampère operator [26]). It hence seems natural to make the following

Conjecture 7.5. *In the case of the ball in \mathbb{C}^n any solution to equation 7.7 is radial and hence given by u_γ above.*

If true the previous conjecture implies the validity of the sharp Moser-Trudinger inequality (without assuming S^1 -invariance), i.e. that \mathcal{G}_γ is bounded in the critical case $\gamma = n + 1$. Indeed, given $u \in \mathcal{H}_0(\mathcal{B})$ we have

$$\mathcal{G}_\gamma(u) = \lim_{\epsilon \rightarrow 0} \mathcal{G}_{\gamma(\epsilon)}(u) \leq \limsup_{\epsilon \rightarrow 0} \mathcal{G}_{\gamma(\epsilon)}$$

But by the previous theorem the sup of $\mathcal{G}_{\gamma(\epsilon)}$ is attained for some function $u_{\gamma(\epsilon)}$ satisfying the equation 7.7, which if the conjecture above is correct has to be radial and thus coincides with ϕ_0^ϵ above. Finally, as shown towards the end in section 3 $\mathcal{G}_a(\phi_0^\epsilon) \rightarrow C_n$ and hence $\mathcal{G}_a(u) \leq C_n$. Note also that by Theorem 1.4 it would be enough to know that any solution is S^1 -invariant in order to deduce the sharp Moser-inequality using the previous argument.

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